Integers represented by positive-definite quadratic forms - the modular approach

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Summary of last time

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- We can write $r_Q(n) = a_E(n) + a_C(n)$. There are explicit lower bounds on $a_E(n)$ of the form $a_E(n) \geq C_E n^{1-\epsilon}$.

- There is a constant C_Q so that $|a_C(n)| \leq C_Q d(n)\sqrt{n}$, but computing C_Q explicitly is hard.

Outline

- Quantitative forms of Tartakowski's theorem

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- Bounding C_Q without computing it.

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- if $r = 4$, n is squarefree.

• Q: For a quaternary form Q, how large is the largest locally represented squarefree n that isn't represented by Q?

Notation (1/2)

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• Let $\|Q\|$ be the largest entry in the matrix A.

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- We write $f(n) \ll n^{k+\epsilon}$ if for all $\epsilon > 0$, $f(n) \leq C_\epsilon n^{k+\epsilon}$ if n is large enough.

Results (1/4)

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If Q is a 4-variable QF and n satisfies appropriate local conditions and $n \gg N(Q)^{14+\epsilon}$, then n is represented by Q.

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Theorem 2 (Browning-Dietmann, 2008)

If Q is a 4-variable QF and n satisfies (different) appropriate local conditions and $n \gg D(Q)^2 \|Q\|^{8+\epsilon}$, then n is represented by Q.

Results (2/4)

- A *discriminant* is an integer $D \equiv 0$ or $1 \pmod 4$. A *fundamental discriminant* D is a discriminant with the property that there is no $k > 1$ so that $k^2 | D$ and $\frac{D}{k^2}$ is a discriminant.

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Theorem 3 (R, 2014)

Suppose that Q is a 4-variable QF and $D(Q)$ is a fundamental discriminant. Then, if $n \gg D(Q)^{2+\epsilon}$, then n is represented by Q.

Results (3/4)

Theorem 4 (R)

Let Q be a 4-variable QF. Assume that $\gcd(n, D(Q)) = 1$ *and n is locally represented by Q. If*

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Theorem 5 (R)

Let Q be a 4-variable QF. Assume that n is locally represented (but not represented by Q) and $n \gg (D(Q)N(Q))^{3+\epsilon}$. Then there is an anisotropic prime p so that $p^2 | n$ and np^{2k} is not represented for any $k \geq 0$.

Results (4/4)

Theorem 6 (R-Thompson)

Suppose that Q is a 4-variable QF and $D(Q) = p$ is prime. Then

$$\sum_{\substack{n \\ r_Q(n)=0}} n \ll p^3.$$

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Theorem 7 (R-Thompson)

Let $p = 8t + 5$ be prime and

$$Q(x, y, z, w) = x^2 + xy + xz + xw + y^2 + yz + yw + z^2 + zw + tw^2.$$

Then $D(Q) = p$ and the largest positive integer not represented by Q is the largest positive integer $m < t$ that is not of the form $4^k(16\ell + 14)$.

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- Instead of exactly computing C_Q, we derive an upper bound for it with less computation. This method works only when $D(Q) = N(Q)$ is a fundamental discriminant.

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• We use the Petersson inner product of two cusp forms $f, g \in S_2(\Gamma_0(D), \chi_D)$ given by

$$\langle f, g \rangle = \frac{3/\pi}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(D)]} \iint_{\mathbb{H}/\Gamma_0(D)} f(x+iy)\overline{g(x+iy)}\, dx\, dy.$$

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• Distinct newforms are orthogonal with respect to the Petersson inner product.

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- From the decomposition of $C(z) = \theta_Q(z) - E(z)$ we get

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$$L(s) = \prod_{p \text{ prime}} \prod_{i=1}^{d} (1 - \alpha_{i,d} p^{-s})^{-1}.$$

- There's a *functional equation* relating $L(s)$ and $L(k - s)$.

Elliptic curve L-functions

- If $E : y^2 = x^3 + Ax + B$, then the L-series of E is $L(E, s) = \sum \frac{a_n(E)}{n^s}$. If p is prime, $a_p(E) = p + 1 - |E(\mathbb{F}_p)|$.

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• If $\Lambda(s) = N(E)^{s/2}(2\pi)^{-s}\Gamma(s)L(E, s)$, then $\Lambda(s) = \epsilon\Lambda(2 - s)$, where $\epsilon \in \{1, -1\}$.

Approximate functional equation

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- This allows one to quickly compute $L(E, 1)$.

Rankin-Selberg L-functions

- If $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=1}^{\infty} b(n)q^n$, the Rankin-Selberg L-function is (approximately)

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^s}.$$

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• The Petersson inner product of f and g is essentially the residue of $L(f \otimes g, s)$ at $s = 1$.

• We require the exact description of local factors of $L(f \otimes g, s)$ and the precise form of the functional equation.

Relation with inner product

- For newforms $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ and $g(z) = \sum_{n=1}^{\infty} b(n) q^n$, we have

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \left(\sum_{\substack{m \mid n \\ n/m \text{ is a square}}} \frac{2^{\omega(n,D)} \mathrm{Re}\,(a(m) b(m))}{m} \right) \frac{1}{n^s}. \qquad (**)$$

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- The residue at $s = 1$ of $L(f \otimes \overline{f}, s)$ is

$$\frac{8\pi^4}{3} \left(\prod_{p|N} 1 + \frac{1}{p} \right) \langle f, f \rangle.$$

Modular forms with complex multiplication

- We say that a newform $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ has *complex multiplication* if there is some discriminant D so that $\chi_D(p) = -1$ implies that $a(p) = 0$.

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• Modular forms with complex multiplication come from Hecke Grössencharacters associated to imaginary quadratic fields.

• Given a discriminant $D(Q)$, it is not difficult to explicitly enumerate the newforms f with complex multiplication in $S_2(\Gamma_0(D(Q)), \chi_{D(Q)})$ and compute $\langle f, f \rangle$.

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- For non-CM f we get

$$\mathrm{Res}_{s=1} L(f \otimes \overline{f}, s) > \frac{1}{26 \log(N)}.$$

Approximate functional equation

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$$\langle f, f \rangle = \frac{1}{N \prod_{p|N}(1 + 1/p)} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,N))}|a(n)|^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

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- Here,

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24x^2 K_0(4\pi x).$$

Extension to arbitrary cusp forms

- If $C_1 = \sum_{i=1}^{s} c_i g_i$ and $C_2 = \sum_{j=1}^{s} d_j g_j$ are two arbitrary cusp forms, define

$$L(C_1 \otimes C_2, s) = \sum_{i=1}^{s} \sum_{j=1}^{s} c_i d_j L(g_i \otimes g_j, s).$$

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• We still have that the residue at $s = 1$ of $L(C_1 \otimes C_2, s)$ is (essentially) $\langle C_1, C_2 \rangle$.

• Is there a simple formula for the coefficients of $L(C_1 \otimes C_2, s)$ in terms of those of C_1 and C_2?

Bilinearity, or lack thereof

• Not in general. We can define two subspaces of $S_2(\Gamma_0(D), \chi_D)$. For $\epsilon \in \{\pm 1\}$, let

$$S_2^{\epsilon}(\Gamma_0(D), \chi_D) = \left\{ \sum c(n) q^n : c(n) = 0 \text{ if } \chi_D(n) = -\epsilon \right\}.$$

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- If $C_1 \in S_2^+$ and $C_2 \in S_2^-$, then $L(C_1 \otimes C_2, s) = 0$ and formula $(**)$ doesn't work.

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• The form Q^* has determinant $D(Q)^3$, level $N(Q)$. Also, $\langle C, C\rangle = D(Q)\langle C^*, C^*\rangle$.

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- The form $C^* \in S_2^-(\Gamma_0(D(Q)), \chi_{D(Q)})$.

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- Factor $\chi_{D(Q)} = \prod_{p|2D(Q)} \chi_p$ as a product of Dirichlet characters with prime power moduli. Let $\epsilon_p(Q)$ be the Hasse invariant of $Q/\mathbb{Q}_p$.

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- Let Q be a positive-definite quaternary form with $D(Q)$ a fundamental discriminant.

- Factor $\chi_{D(Q)} = \prod_{p|2D(Q)} \chi_p$ as a product of Dirichlet characters with prime power moduli. Let $\epsilon_p(Q)$ be the Hasse invariant of $Q/\mathbb{Q}_p$.

- Show that if $p|2D(Q)$ is an odd prime and m is a positive integer coprime to p represented by Q^*, then $\chi_p(m) = \epsilon_p(Q)$. Show that if m is an odd integer represented by Q^*, then $\chi_2(m) = -\epsilon_2(Q)$. Conclude that if m is represented by Q^*, then either $\chi_D(m) = 0$ or $\chi_D(m) = -1$.

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- We can use formula $(**)$ to estimate $\langle C^*, C^* \rangle$.
- We find a number B so that $\langle g, g \rangle \geq B$ for all newforms $g \in S_2$.
- We get that

$$C_Q \leq \sqrt{\frac{D(Q)\langle C^*, C^* \rangle \dim S_2}{B}}.$$

Example (1/2)

- For

$$Q(x, y, z, w) = x^2 + 3y^2 + 3yz + 3yw + 5z^2 + zw + 34w^2$$

we have $D(Q) = 6780$.

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- The space $S_2(\Gamma_0(6780), \chi_{6780})$ has four Galois-orbits of newforms of sizes 4, 4, 40, and 1312.

- We find that for all newforms g,

$$\langle g, g \rangle \geq 1.019 \cdot 10^{-5}.$$

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• This gives $C_Q \leq 1199.86$. It follows that Q represents every odd number larger than $8.315 \cdot 10^{16}$. These computations take 3 minutes and 50 seconds.

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• We compute the first 101700 coefficients of $\theta_{Q^*}(z)$ and $E^*(z)$. We use this to find that

$$0.01066 \leq \langle C, C \rangle \leq 0.01079.$$

• This gives $C_Q \leq 1199.86$. It follows that Q represents every odd number larger than $8.315 \cdot 10^{16}$. These computations take 3 minutes and 50 seconds.

• Checking up to this bound requires 22 minutes and 29 seconds. We find that Q represents all odd numbers.

Overview of proof

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- Next, I'll give an overview of the proof of Theorem 3, which states that if $D(Q)$ is a fundamental discriminant, and n is locally represented by Q with $n \gg D(Q)^{2+\epsilon}$, then n is represented.

Proof of Theorem 3 - Eisenstein part

• Let Q be a quaternary form with $D(Q)$ a fundamental discriminant. Recall that $r_Q(n) = a_E(n) + a_C(n)$.

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• The form Q is not anisotropic at any prime. Also,

$$a_E(n) \gg \frac{n^{1-\epsilon}}{\sqrt{D(Q)}}$$

if n is locally represented.

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- We can give an *ineffective* lower bound $B \gg D(Q)^{-\epsilon}$.

Proof of Theorem 3 - Petersson norm

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- Therefore,

$$\langle C, C \rangle = \frac{D(Q)}{\sigma(D(Q))} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n, D(Q)))} a_{C^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d \sqrt{\frac{n}{D(Q)}}\right).$$

Claim: $\langle C, C \rangle \ll 1$

- We have $a_{C^*}(n) = r_{Q^*}(n) - a_{E^*}(n)$ and so $a_{C^*}(n)^2 \ll r_{Q^*}(n)^2 + a_{E^*}(n)^2$. The first term is much bigger than the second.

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- The terms with $n \ll D(Q)\log^2(D(Q))$ are basically

$$\sum_{n=1}^{cD(Q)\log^2(D(Q))} \frac{r_{Q^*}(n)^2}{n}.$$

Trick

- Using partial summation, we can write this as

$$\int_1^{\infty} \frac{1}{t^2} \left(\sum_{n \leq \min(t, cD(Q) \log^2(D(Q)))} r_{Q^*}(n)^2 \right) dt.$$

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- The best way to bound $\sum_{n \leq t} r_{Q^*}(n)^2$ is to use the inequality

$$\sum_{n \leq t} r_{Q^*}(n)^2 \leq \left(\sum_{n \leq t} r_{Q^*}(n) \right) \cdot \max_{n \leq t} r_{Q^*}(n).$$

Result

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- We have

$$\max_{n \leq t} r_{Q^*}(n) \ll \begin{cases} 1 & x \leq D(Q)^{1/2} \\ \frac{x^{1/2}}{D(Q)^{1/4}} & D(Q)^{1/2} \leq x \leq D(Q)^{5/6} \\ \frac{x}{D(Q)^{2/3}} & D(Q)^{5/6} \leq x \leq D(Q)^{11/12} \\ \frac{x^{3/2}}{D(Q)^{9/8}} & x \geq D(Q)^{11/12}. \end{cases}$$

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- Not only that, the main contribution to $\langle C, C \rangle$ comes from very small n (like $n \ll D(Q)^{\epsilon}$).

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- Hence, $|a_C(n)| \ll D(Q)^{1/2+\epsilon} n^{1/2+\epsilon}$.

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Summary of proof (2/2)

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- Thus,
$$r_Q(n) \gg \frac{n^{1-\epsilon}}{\sqrt{D(Q)}} - D(Q)^{1/2+\epsilon} n^{1/2+\epsilon}.$$
- If $n \gg D(Q)^{2+\epsilon}$, $r_Q(n) > 0$ and n is represented by Q.

Thank you!

- Suppose Q is a positive-definite, quaternary quadratic form, and $D(Q)$ is a fundamental discriminant. If Q locally represents $n \gg D(Q)^{2+\epsilon}$, then n is represented by Q.

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- No good generalization of this result is known for forms with $D(Q)$ not a fundamental discriminant.

- More details can be found in the paper at `http://arxiv.org/abs/1111.0979`.