BOUNDS FOR THE COEFFICIENTS OF POWERS OF THE Δ-FUNCTION

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Abstract. For $k \geq 1$, let $\sum_{n=k}^{\infty} \tau_k(n) q^n = q^k \prod_{n=1}^{\infty} (1-q^n)^{24k}$. It follows from Deligne's proof of the Weil conjectures that there is a constant C_k so that $|\tau_k(n)| \leq C_k d(n) n^{(12k-1)/2}$. We study the value of C_k as a function of k, and show that it tends to zero very rapidly.

1. Introduction and Statement of Results

For an integer r, define the numbers $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^r.$$

For various values of r, these numbers capture important arithmetic objects. For example, when $r = -1$, we recover the classical partition generating function

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

while for $r = 1$ and $r = 3$ we recover the identities of Euler and Jacobi,

$$\sum_{n=0}^{\infty} p_1(n) q^n = \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}$$

$$\sum_{n=0}^{\infty} p_3(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(n^2+n)/2}.$$

In a series of papers ([9], [10], [11], [12]), Newman studied the function $p_r(n)$, and proved a number of identities for it. Newman was particularly interested in when the function $p_r(n)$ is zero and computed $p_r(n)$ for small n (as a polynomial in r). These coefficients were later considered by many authors, including Gupta, Atkin, Costello, Gordon, and finally Serre. Serre showed in [16] that if r is an even integer, $\{n : p_r(n) = 0\}$ has density zero if and only if $r = 2, 4, 6, 8, 10, 14$, or 26.

Another natural question is about how large (as a function of r and n) the coefficients $p_r(n)$ are. In this regard, Newman's approach of expressing the coefficients

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$p_r(n)$ as polynomials in r is very ineffective. A stronger result follows from work of Deligne [2] (at least when r is even) and gives that $p_r(n) \ll n^{(r-1)/2+\epsilon}$. In the case of $r = 24$, it implies Ramanujan's famous conjecture that if

$$\sum_{n=1}^{\infty} p_{24}(n-1) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

is the Fourier expansion of the weight 12 cusp form $\Delta(z)$, then

$$|p_{24}(n-1)| \leq d(n) n^{11/2},$$

where $d(n)$ is the number of divisors of n.

Deligne's bound applies to cuspidal Hecke eigenforms of all weights. Hence, if $f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_m$ is any cusp form of weight m, then by writing $f(z) = \sum_{i=1}^{\dim S_m} c_i f_i$, where the f_i are normalized Hecke eigenforms, we have that $|a(n)| \leq C d(n) n^{(m-1)/2}$, where $C = \sum_{i=1}^{\dim S_m} |c_i|$.

For example, we may write

$$\Delta^2(z) = \sum_{n=2}^{\infty} p_{48}(n-2) q^n = q^2 - 48q^3 + 1080q^4 + \cdots \in S_{24}$$

as a linear combination of the Hecke eigenforms

$$f_1(z) = q + (540 + 12\sqrt{144169}) q^2 + (169740 - 576\sqrt{144169}) q^3 + \cdots$$
$$f_2(z) = q + (540 - 12\sqrt{144169}) q^2 + (169740 + 576\sqrt{144169}) q^3 + \cdots.$$

We have then that

$$\Delta^2(z) = \frac{f_1 - f_2}{24\sqrt{144169}},$$

and hence $|p_{48}(n-2)| \leq \frac{1}{12\sqrt{144169}} d(n) n^{23/2}$. Note that $\frac{1}{12\sqrt{144169}} \approx 0.000219$ is quite small.

The goal of this paper is to compute explicit bounds for the coefficients $p_r(n)$, when $r \geq 0$ and is a multiple of 24. We then have that

$$\Delta^k(z) := \sum_{n=k}^{\infty} p_{24k}(n-k) q^n.$$

Let $C_k := \sum_{i=1}^{k} |c_i|$, where $\Delta^k(z) = \sum_{i=1}^{k} c_i f_i$ is the representation of Δ^k as a sum of Hecke eigenforms. Then,

$$|p_{24k}(n-k)| \leq C_k d(n) n^{(12k-1)/2}.$$

It suffices therefore to bound C_k. Our main result is the following.

Theorem 1. *For $k \geq 2$, we have*

$$\log(C_k) = -6k\log(k) + 6k\log\left(\frac{2\pi^3 e}{27\Gamma(2/3)^6}\right) + O(\log(k)).$$

This result follows from explicit upper and lower bounds on C_k derived below. Our approach is the following. For $f, g \in S_k$, let

$$\langle f, g\rangle_k = \frac{3}{\pi}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(x+iy)\overline{g(x+iy)}y^k\,\frac{dx\,dy}{y^2}$$

be the normalized Petersson inner product of f and g. Elementary considerations provide bounds on $\langle\Delta^k, \Delta^k\rangle_{12k}$. If $f_i \in S_{12k}$ is a normalized Hecke eigenform, then $\langle f_i, f_i\rangle_{12k}$ is essentially the special value at $s = 1$ of the symmetric square L-function associated to f_i. Goldfeld, Hoffstein, and Lieman showed in the appendix to [5], that such an L-function can have no Siegel zero. We make their argument explicit and derive an explicit lower bound on $\langle f_i, f_i\rangle_{12k}$.

These bounds are translated to bounds on C_k using the well-known fact (see Theorem 6.12 of [6]) that if $f_i \neq f_j$ are Hecke eigenforms, then $\langle f_i, f_j\rangle_{12k} = 0$.

Remark. *It is plausible that in fact,*

$$C_k = \sup_{n\geq 1}\frac{|p_{24k}(n-k)|}{d(n)n^{(12k-1)/2}}.$$

This would follow if for each eigenform $f_i = \sum_{n=1}^{\infty} a_i(n)q^n$, we have $|a_i(p)| \geq (2-\epsilon)p^{(12k-1)/2}$ for a positive density set of primes, and if the coefficients $a_1(p), a_2(p), \ldots, a_k(p)$ are "independent." The first statement would follow from the Sato-Tate conjecture. Recently, Richard Taylor has achieved an important breakthrough by proving the Sato-Tate conjecture for a wide class of elliptic curves. Taylor's work establishes the automorphy of symmetric power L-functions, which can be used (as in [14]*) to produce lower bounds for Hecke eigenvalues.*

Remark. *The approach given here readily generalizes to powers of any fixed modular form, provided the powers are orthogonal to CM forms. One cannot (at present) exclude the possible existence of a Siegel zero for the symmetric square of a CM form. For $r \equiv 0, 12, 16 \pmod{24}$, $\sum p_r(n)q^n$ can be related to a modular form lying in a space with no CM forms.*

In Section 2 we derive upper and lower bounds on the Petersson norms $\langle\Delta^k, \Delta^k\rangle_{12k}$ and $\langle f_i, f_i\rangle_{12k}$. In Section 3 we use the results derived in Section 2 to prove Theorem 1, and in Section 4 we present some numerical data.

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2. Petersson norm bounds

First, we will compute bounds for the Petersson norm of Hecke eigenforms $f_i \in S_{12k}$. We will repeatedly use the fact (see the second equation on pg. 251 of [6]) that

$$L(\mathrm{Sym}^2 f_i, 1) = \frac{6}{\pi^2} \cdot \frac{(4\pi)^{12k} \langle f_i, f_i \rangle_{12k}}{\Gamma(12k)}.$$

If the normalized L-function of $f_i = \sum_{n=1}^{\infty} a_i(n) q^n$ is

$$L(f_i, s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1},$$

where $\alpha_p + \beta_p = a_i(p)/p^{(12k-1)/2}$ and $\alpha_p \beta_p = 1$, then

$$L(\mathrm{Sym}^2 f_i, s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}.$$

This L-function is known by work of Gelbart and Jacquet [3] to be the L-function of a cuspidal automorphic representation on $GL(3)$. Hence, it is entire and if

$$\Lambda(\mathrm{Sym}^2 f_i, s) = \pi^{-3s/2} \Gamma((s+1)/2) \Gamma((s + (12k-1))/2) \Gamma((s+12k)/2) L(\mathrm{Sym}^2 f_i, s),$$

then $\Lambda(\mathrm{Sym}^2 f_i, s) = \Lambda(\mathrm{Sym}^2 f_i, 1-s)$.

Lemma 2. *If $f_i \in S_{12k}$ is a normalized Hecke eigenform, then*

$$L(\mathrm{Sym}^2 f_i, s) \neq 0$$

for $s > 1 - \frac{5 - 2\sqrt{6}}{10 \log(12k)}$.

Proof. Goldfeld, Hoffstein and Lieman introduce the auxiliary function

$$L(s) = \zeta(s)^2 L(\mathrm{Sym}^2 f_i, s)^3 L(\mathrm{Sym}^4 f_i, s).$$

Here,

$$L(\mathrm{Sym}^4 f_i, s) = \prod_p (1 - \alpha_p^4 p^{-s})^{-1} (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \alpha_p^{-2} p^{-s})^{-1} (1 - \alpha_p^{-4} p^{-s})^{-1}.$$

Work of Kim [7] implies that this is the L-function of a cuspidal automorphic representation on $GL(5)$. From this, it follows that $L(\mathrm{Sym}^4 f_i, s)$ has an analytic continuation and functional equation of the usual type (see the paper of Cogdell and Michel [1] for details about computing the sign of the functional equation and the Γ-factors of symmetric power L-functions using the local Langlands correspondence for $GL(n)$).

If we let $\Lambda(s) = s^2 (1-s)^2 G(s) L(s)$, where

$$G(s) = \pi^{-16s/2} \Gamma(s/2)^3 \Gamma((s+1)/2)^3 \Gamma((s + (12k-1))/2)^4$$
$$\Gamma((s+12k)/2)^4 \Gamma((s + (24k-2))/2) \Gamma((s + (24k-1))/2),$$

then $\Lambda(1-s) = \Lambda(s)$. Writing $\Lambda(s) = e^{A+Bs}\prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$ and taking the logarithmic derivative gives

$$\sum_\rho \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{2}{s} + \frac{2}{1-s} + \frac{L'(s)}{L(s)} + \frac{G'(s)}{G(s)} - B.$$

Now, the Dirichlet coefficients of $L(s)$ are non-negative. This implies that for $\mathrm{Re}\,(s) > 1$, $L'(s)/L(s) < 0$. Taking the real part of this equation and noting that $\mathrm{Re}\,(B) = -\sum_\rho \mathrm{Re}\,(1/\rho)$ gives

$$\sum_\rho \mathrm{Re}\left(\frac{1}{s-\rho}\right) \le \frac{2}{s} + \frac{2}{1-s} + \frac{G'(s)}{G(s)}.$$

Assume that $s = 1+\alpha$, where $0 < \alpha \le 1/2$ will be chosen later. Noting that $\Gamma'(s)/\Gamma(s) \le \log(s)$ for $s \ge 1$ gives that in this range, $G'(s)/G(s) \le 10\log(12k) - 2$.

Suppose that $L(\mathrm{Sym}^2 f, \beta) = 0$. Then, we have

$$\frac{3}{\alpha + 1 - \beta} \le \frac{2}{\alpha} + 10\log(12k).$$

Solving for β and choosing α optimally yields the desired result. □

Next, we follow the argument of Hoffstein in [4] to translate this into an explicit lower bound on $L(\mathrm{Sym}^2 f_i, 1)$.

Lemma 3. *If $f \in S_{12k}$ is a normalized Hecke eigenform, then*

$$L(\mathrm{Sym}^2 f, 1) > \frac{1}{64\log(12k)}.$$

Proof. Let

$$L(f\otimes f, s) = \zeta(s) L(\mathrm{Sym}^2 f, s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}.$$

Then, $a(n) \ge 0$ for all $n \ge 1$. Also, its functional equation is well-known (for example, it follows from that of $L(\mathrm{Sym}^2 f, s)$).

Let $\beta = 1 - \frac{5-2\sqrt{6}}{10\log(12k)}$. We set $x = (12k)^A$. It will turn out that the optimal A is about $8/5$ and we choose $A = \frac{8}{5} + \frac{10}{\log(12k)}$. We consider

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L(f\otimes f, s+\beta) x^s\, ds}{s\prod_{k=2}^{10}(s+k)}.$$

We use the fact that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s\, ds}{s\prod_{k=2}^{10}(s+k)} = \begin{cases} \frac{(x+9)(x-1)^9}{10!x^{10}} & x > 1 \\ 0 & x < 1, \end{cases}$$

and conclude that

$$I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L(f \otimes f, s+\beta) x^s}{s \prod_{k=2}^{10} (s+k)} = \sum_{n \leq x} \frac{a(n) \left(\frac{x}{n} + 9\right) \left(\frac{x}{n} - 1\right)^9}{10! n^\beta (x/n)^{10}}.$$

One can easily show that $a(n^2) \geq 1$. We consider only those terms for which $x/n \geq 559$. This gives a lower bound on the integral of $\frac{1.6442234}{10!}$.

Now, we move the contour to $\mathrm{Re}\,(s) = \alpha$, where $\alpha = -3/2 - \beta$. We pick up poles at $s = 1 - \beta$, $s = 0$ and $s = -2$. This gives

$$I = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{L(f \otimes f, s+\beta) x^s \, ds}{s \prod_{k=2}^{10} (s+k)} + \frac{L(\mathrm{Sym}^2 f, 1) x^{1-\beta}}{(1-\beta) \prod_{k=2}^{10} (1 - \beta + k)} + \frac{L(f \otimes f, \beta)}{10!} + \frac{L(f \otimes f, -2+\beta) x^{-2}}{2 \cdot 8!}.$$

There are no zeroes of $L(\mathrm{Sym}^2 f, s)$ to the right of β and hence $L(\mathrm{Sym}^2 f, \beta) \geq 0$. Since $\zeta(\beta) < 0$, it follows that $L(f \otimes f, \beta) \leq 0$. Also, $L(f \otimes f, -2+\beta) < 0$. It follows that

$$(1) \qquad \frac{1.6442234}{10!} - \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{L(f \otimes f, s+\beta) x^s \, ds}{s \prod_{k=2}^{10} (s+k)} \leq \frac{L(\mathrm{Sym}^2 f, 1) x^{1-\beta}}{(1-\beta) \prod_{k=2}^{10} (1 - \beta + k)}.$$

Now, we bound the integral in the above inequality. The functional equation for $L(f \otimes f, s)$ implies that

$$\frac{|L(f \otimes f, -3/2 + it)|}{|L(f \otimes f, 5/2 - it)|} = |1/2 + it|^2 |3/2 + it|^2 \prod_{m=1}^{4} |12k - 3 + m/2 + it|.$$

Also, $|L(f \otimes f, 5/2 - it)| \leq \zeta(5/2)^4$. Hence, $|I|$ is bounded above by

$$\frac{\zeta(5/2)^4}{2^9 \pi^9} (12k)^{A(-3/2-\beta)} \cdot \int_{-\infty}^{\infty} \frac{|1/2 + it|^2 |3/2 + it|^2 \prod_{m=1}^{4} |12k - 3 + m/2 + it| \, dt}{|9/4 + it| |1/4 + it| \prod_{n=3}^{\infty} |n - 5/2 + it|}$$
$$\leq \frac{\zeta(5/2)^4 (12k)^{4 - A(3/2+\beta)}}{2^9 \pi^9} \int_{-\infty}^{\infty} \frac{|1/2 + it| |3/2 + it| |1 + it|^3 |25/24 + it|}{|1/4 + it| |9/4 + it| \prod_{n=2}^{7} |n + 1/2 + it|}$$
$$\leq \frac{(12k)^{4 - A(3/2+\beta)} \cdot .181266}{10!}.$$

Hence, returning to equation (1), we have

$$L(\mathrm{Sym}^2 f, 1) \geq (1 - \beta) \left(\frac{1.6442234}{(12k)^{A(1-\beta)}} - \frac{.181266}{(12k)^{(5/2)A - 4}} \right).$$

We choose $A = \frac{8}{5} + \frac{10}{\log(12k)}$ and obtain the desired result. ☐

Next, we use an elementary argument to obtain an upper bound for $\langle f_i, f_i\rangle_{12k}$.

Lemma 4. *If f_i is a normalized Hecke eigenform of weight k and $k \geq 48$, then*

$$\langle f_i, f_i\rangle_k \leq 3.182 \frac{\Gamma(k)\log^3(k)}{(4\pi)^k}.$$

Remark. *This result could also be obtained from the convexity bound for $L(\mathrm{Sym}^2 f_i, s)$.*

Proof. For brevity we will explain only the main ideas. One can extend the integral for $\langle f_i, f_i\rangle_k$ to the region $\{x+iy : -1/2 \leq x \leq 1/2, y \geq \sqrt{3}/2\}$. If

$$f_i(z) = \sum_{n=1}^{\infty} a_i(n) q^n,$$

and we replace $f_i(z)$ by its Fourier expansion, then we obtain the upper bound

$$\langle f_i, f_i\rangle_k \leq \frac{3}{\pi} \sum_{n=1}^{\infty} |a_i(n)|^2 \int_{\sqrt{3}/2}^{\infty} e^{-4\pi n y} y^{k-2}\, dy.$$

Changing variables, we get

$$\frac{1}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{|a_i(n)|^2}{n^{k-1}} \int_{\pi\sqrt{3}n}^{\infty} e^{-u} u^{k-2}\, du.$$

The Deligne bound implies that $|a_i(n)|^2/n^{k-1} \leq d(n)^2$. The integrand also does not depend on n. Replacing the order of the sum and the integral we obtain

$$\frac{1}{(4\pi)^{k-1}} \int_{\pi\sqrt{3}}^{\infty} u^{k-2} e^{-u} \sum_{n\leq u} d(n)^2\, du.$$

An asymptotic for $\sum_{n\leq u} d(n)^2$ was given by Ramanujan ([13], equation (B)). The elementary proof of $\sum_{n\leq u} d(n)^2 \sim \frac{1}{\pi^2} u \log^3(u)$ in ([8], Theorem 7.8) can be easily modified to show that $\sum_{n\leq u} d(n)^2 \leq \frac{19}{3\log^3(6)} u \log^3(u)$ for all $u \geq 1$. Hence, it suffices to estimate

$$\int_{\pi\sqrt{3}}^{\infty} e^{-u} u^{k-2} \log^3(u)\, du.$$

One can easily check that the integrand decays rapidly for $u \gg k\log(k)$. The remainder is easy to estimate by comparison with the Γ-function. □

The next result is of independent interest and is useful in bounding $\langle \Delta^k, \Delta^k\rangle_{12k}$.

Lemma 5. *Let $f(x,y) = |\Delta(x+iy)|^2 y^{12}$. Then, for $y > 0$ we have*

$$f(x,y) \leq B := \left(\frac{\sqrt{2}\pi}{3\Gamma(2/3)^3}\right)^{24}$$

with equality if and only if $x + iy = \frac{a\omega+b}{c\omega+d}$ *for* $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc = 1$ *and* $\omega = \frac{-1+i\sqrt{3}}{2}$.

Proof. First, the equality when $x = -1/2$ and $y = \sqrt{3}/2$ is very classical (see for example equation 2 on pg. 110 of [15]).

Next, the function $|\Delta(z)|^2 \mathrm{Im}\,(z)^{12}$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. It suffices therefore to find its maximum on the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, namely $\{z \in \mathbb{C} : -1/2 \leq \mathrm{Re}\,(z) \leq 1/2 \text{ and } |z| \geq 1\}$. Moreover, since the Fourier coefficients of $\Delta(z)$ are real, it follows that $\overline{\Delta(x+iy)} = \Delta(-x+iy)$. Thus, $f(x,y) = f(1-x,y)$ and it suffices to consider $-1/2 \leq x \leq 0$.

We approximate the size of $|\Delta(x+iy)|$ by $\left|\sum_{n=1}^{4} \tau(n) q^n\right|$. We can easily see that for any y this is maximized when $x = -1/2$. One can also show that $y^6 \left|\sum_{n=1}^{4} \tau(n) q^n\right|$ is maximized when $y = \sqrt{3}/2$. It follows from this that if $f(x,y) \geq f(-1/2, \sqrt{3}/2)$ for $x + iy$ in the fundamental domain, then $y \leq 0.8676$ and hence $-1/2 \leq x \leq -0.497$.

Differentiating the equality $f(x,y) = f(1-x,y)$ with respect to x and setting $x = -1/2$ shows that $f_x(-1/2, y) = 0$ for all y. Using the transformation law with the matrix $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ shows that

$$f\left(\frac{-x^2 - y^2 - x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = f(x,y).$$

Differentiating this with respect to x, setting $x = -1/2$, $y = \sqrt{3}/2$ and using that $f_x(-1/2, \sqrt{3}/2) = 0$ shows that $f_y(-1/2, \sqrt{3}/2) = 0$. Since the maximum of $f(x,y)$ occurs where f_x and f_y both vanish, it suffices to show that this does not occur elsewhere in the box $-1/2 \leq x \leq -0.497$, $\sqrt{3}/2 \leq y \leq 0.8676$.

Next, we use the product expansion $f(x,y) = y^{12} \prod_{n=1}^{\infty} |1 - q^n|^{48}$. This implies that

$$\frac{f_x}{f} = 24 \sum_{n=1}^{\infty} \frac{2\pi n \sin(2\pi n x) e^{-2\pi n y}}{1 - 2\cos(2\pi n x) e^{-2\pi n y} + e^{-4\pi n y}}.$$

We note that

$$\frac{f_{xx}}{f} = \frac{d}{dx}\left(\frac{f_x}{f}\right) + \left(\frac{f_x}{f}\right)^2.$$

Trivially estimating f_x/f, we see that $|f_x/f| \leq 0.665$ in this box. We estimate all but the first two terms of $\frac{d}{dx}(f_x/f)$ trivially and obtain the bound $f_{xx}/f \leq -1.9$.

Now, we assume $x = -1/2$. Using

$$\frac{f_y}{f} = \frac{12}{y} - 4\pi + 96\pi \sum_{n=1}^{\infty} \frac{(-1)^n n e^{-2\pi n y}}{1 - (-1)^n e^{-2\pi n y}},$$

we will estimate

$$\frac{f_{yy}}{f} = \frac{d}{dy}\left(\frac{f_y}{f}\right) + \left(\frac{f_y}{f}\right)^2.$$

We see that $|f_y/f| \leq 0.048$. The main term is $-12/y^2$, and for $y \leq 1.1$, this dominates and $f_{yy}/f < 0$. This establishes the desired result since we have $f_x < 0$ for $x \neq -1/2$ and if $x = -1/2$ we have $f_x = 0$ and $f_y < 0$ unless $y = \sqrt{3}/2$. □

With a little bit of work, the above lemma can be translated into bounds on $\langle \Delta^k, \Delta^k \rangle_{12k}$.

Lemma 6. *For $k \geq 1$, we have*

$$\frac{0.08906 B^k}{k} \leq \langle \Delta^k, \Delta^k \rangle_{12k} \leq \frac{76.4 B^k}{k}.$$

Proof. For the lower bound, similar arguments to those in the proof of Lemma 5 imply that for all x and y, $f_{xx} \geq -4.251 \cdot f$ and for $x = -1/2$, $f_{yy} \geq -8.652 \cdot f$. Using the upper bound on f established above, we obtain that if $C := -3.555 \cdot 10^{-5}$, then $f_{xx} \geq C$ for all x and y and $f_{yy} \geq C$ for $x = -1/2$ and $y \geq \sqrt{3}/2$. Integrating from $(-1/2, \sqrt{3}/2)$ to $(-1/2, y)$ and then to (x, y) shows that

$$f(x,y) - f(-1/2, \sqrt{3}/2) \leq -(C/2)((x+1/2)^2 + (y - \sqrt{3}/2)^2).$$

If we fix $\epsilon > 0$ it follows that $f(x,y) \geq B - \epsilon$ on a set of measure at least $\frac{2\pi}{3(3.555 \cdot 10^{-5})}\epsilon$. This gives a lower bound for the Petersson norm of

$$\frac{2\pi}{3(3.555 \cdot 10^{-5})}\epsilon(B-\epsilon)^k.$$

This is maximized with $\epsilon = \frac{B}{k+1}$ and this gives the desired result.

For the upper bound, we let

$$S_\epsilon = \{(x,y) : -1/2 \leq x \leq 1/2, x^2 + y^2 \geq 1, \text{ and } f(x,y) \leq B - \epsilon\}.$$

One can check that there is a constant C_2 so that if ϵ is small enough then $\mu(S_\epsilon) \leq C_2\epsilon$, where $\mu = \frac{3}{\pi}\frac{dx\,dy}{y^2}$. Choose ϵ small enough so that $\mu(S_\delta) \leq C_2\delta$ for all $\delta \leq \epsilon$ and let n be a positive integer. For $(x,y) \in S_{(l+1)\epsilon/n} - S_{l\epsilon/n}$, we have $f(x,y) \leq B - \frac{\epsilon l}{n}$. It follows that

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leq \sum_{l=0}^{n-1} \mu\left(S_{(l+1)\epsilon/n} - S_{l\epsilon/n}\right)\left(B - \frac{\epsilon l}{n}\right)^k + \frac{3}{\pi}(B-\epsilon)^k.$$

Let $a_l = \mu\left(S_{(l+1)\epsilon/n} - S_{l\epsilon/n}\right)$ and $b_l = \left(B - \frac{\epsilon l}{n}\right)^k$. Notice that $b_0 \geq b_1 \geq \cdots \geq b_{n-1}$. Also for $0 \leq m \leq n-1$,

$$\sum_{l=0}^{m} a_l = \mu\left(S_{(m+1)\epsilon/n}\right) \leq C_2(m+1)\epsilon/n.$$

Since the b_l are decreasing, the sum $\sum_{l=0}^{n-1} a_l b_l \leq \sum_{l=0}^{n-1} (C_2\epsilon/n) b_l$. It follows that

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leq C_2 \sum_{l=0}^{n-1} \frac{\epsilon}{n} \left(B - \frac{\epsilon l}{n} \right)^k + \frac{3}{\pi}(B-\epsilon)^k.$$

Taking the limit as $n \to \infty$ gives that the first term above is

$$\int_0^\epsilon C_2 (B-x)^k \, dx = \frac{C_2}{k+1} \left[B^{k+1} - (B-\epsilon)^{k+1} \right].$$

Hence, we have

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leq \frac{C_2 B^{k+1}}{k+1} \left[1 - (1-\epsilon/B)^{k+1} + \frac{3(k+1)}{\pi B C_2} (1-\epsilon/B)^k \right].$$

Computations similar to those above show that we may take $\epsilon = 1.93553 \cdot 10^{-8}$, and $C_2 = 729582$. This gives that for $k \geq 1$,

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leq \frac{76.4 B^k}{k}.$$

Note however, that for small k and for $k \geq 300$ the above inequality is better. $\square$

3. Proof of Theorem 1

Proof. We assume that $k \geq 4$ and write

$$\Delta^k = \sum_{i=1}^k c_i f_i,$$

where the f_i are normalized Hecke eigenforms. Since the Fourier coefficients of the f_i are real, the c_i are real. As noted in the introduction, if $i \neq j$, then $\langle f_i, f_j \rangle = 0$. Computing the inner product of Δ^k with itself, we obtain

$$\langle \Delta^k, \Delta^k \rangle_{12k} = \sum_{i=1}^k c_i^2 \langle f_i, f_i \rangle_{12k}.$$

Let B_1 and B_2 be the lower and upper bounds on $\langle f_i, f_i \rangle_{12k}$ furnished by Lemmas 3 and 4, respectively. We obtain

$$\frac{\langle \Delta^k, \Delta^k \rangle_{12k}}{B_2} \leq \sum_{i=1}^k c_i^2 \leq \frac{\langle \Delta^k, \Delta^k \rangle_{12k}}{B_1}.$$

We use Lemma 6 together with the simple inequalities

$$\sqrt{\sum_{i=1}^k c_i^2} \leq \sum_{i=1}^k |c_i| \leq \sqrt{k} \sqrt{\sum_{i=1}^k c_i^2}$$

to complete the proof. This gives the explicit bound

$$\frac{(4\pi)^{6k}B^{k/2}}{6\sqrt{(12k-1)!}\sqrt{k}\log^{3/2}(12k)} \leq C_k \leq \frac{55(4\pi)^{6k}B^{k/2}\log^{1/2}(12k)}{\sqrt{(12k-1)!}}$$

Taking logarithms easily yields the desired result. □

4. Appendix - Numerical Data

Using MAGMA, if k is small, we can compute the Fourier expansions of the normalized Hecke eigenforms f_i and hence compute $C_k = \sum_{i=1}^{k} |c_i|$. The table below is a list of k and the logarithms of the bounds derived in this paper.

k	log(lower bound)	$\log(C_k)$	log(upper bound)
1	−2.9232	0.0000	2.7527
2	−11.706	−8.4243	−4.8448
3	−23.369	−19.657	−15.862
4	−36.977	−33.072	−29.028
5	−52.053	−47.874	−43.769
6	−68.308	−64.102	−59.754
7	−85.549	−81.120	−76.771
8	−103.64	−99.160	−94.665
9	−122.46	−117.84	−113.33
10	−141.96	−137.40	−132.66

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