ZAGIER DUALITY FOR THE EXPONENTS OF BORCHERDS PRODUCTS FOR HILBERT MODULAR FORMS

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Abstract. A certain sequence of weight 1/2 modular forms arises in the theory of Borcherds products for modular forms for $SL_2(\mathbb{Z})$. Zagier proved a family of identities between the coefficients of these weight 1/2 forms and a similar sequence of weight 3/2 modular forms, which interpolate traces of singular moduli. We obtain the analogous results for modular forms arising from Borcherds products for Hilbert modular forms.

1. Introduction and Statement of Results

For an integer $k \geq 0$, let $M^{+}_{k+1/2}(\Gamma_0(4))$ denote the vector space of nearly holomorphic modular forms $f(z)$ of weight $k+1/2$ for $\Gamma_0(4)$ with the property that if

$$f(z) = \sum_{n=-h}^{\infty} c(n) q^n$$

is the Fourier expansion of f in the variable $q = e^{2\pi i z}$, then $c(n) = 0$ if $(-1)^k n \equiv 2, 3 \pmod 4$. Here a modular form is called nearly holomorphic if it is holomorphic on the upper half plane and meromorphic at the cusps. For $d \geq 0$, $d \equiv 0, 3 \pmod 4$ let $f_d(z)$ denote the unique form in $M^{+}_{1/2}(\Gamma_0(4))$ with

$$f_d(z) = q^{-d} + O(q) = \sum_{n=-d}^{\infty} a_d(n) q^n. \tag{1}$$

The $f_d(z)$ form a basis for the space $M^{+}_{1/2}(\Gamma_0(4))$. As proven by Borcherds in [1] (Theorem 14.1), the $f_d(z)$ have an interpretation in terms of infinite product expansions of certain meromorphic modular forms for $SL_2(\mathbb{Z})$.

Theorem (Borcherds). *Let $f(z) = \sum c(n) q^n \in M^{+}_{1/2}(\Gamma_0(4))$ with $c(n) \in \mathbb{Z}$. If*

$$\psi(z) = q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{c(n^2)},$$

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where h is the constant term of $f(z)\sum_n H(n)q^n$ (here $H(n)$ is the Hurwitz class number), then $\psi(z)$ has the following properties:

1. *The function $\psi(z)$ is a meromorphic modular form of weight $c(0)$ for some character of $SL_2(\mathbb{Z})$, with leading coefficient 1 and integer coefficients.*
2. *All zeroes and poles of ψ are at imaginary quadratic irrationals. Moreover, the multiplicity of zero or pole at an imaginary quadratic irrationality τ of discriminant $D < 0$ is*
$$\sum_{d>0} c(Dd^2).$$

Conversely, any meromorphic modular form $\psi(z)$ for $SL_2(\mathbb{Z})$ with zeroes and poles at imaginary quadratic irrationals has a product representation of the above form.

Here, if τ is the root in the upper half plane $\mathbb{H}$ of $az^2+bz+c=0$ with $\gcd(a,b,c)=1$, then we say the discriminant of τ is b^2-4ac. For $d \geq 1$, $d \equiv 0, 1 \pmod 4$, let $g_d(z)$ denote the unique form in $M^+_{3/2}(\Gamma_0(4))$ with

$$g_d(z) = q^{-d} + O(1) = \sum_{n=-d}^{\infty} b_d(n)q^n. \tag{2}$$

In [5] (Theorem 4), Zagier proves the "duality" theorem relating the forms $f_d(z)$ and $g_d(z)$.

Theorem (Zagier)**.** *If d is a non-negative integer, $n \geq 1$ with $d \equiv 0, 3 \pmod 4$ and $n \equiv 0, 1 \pmod 4$, then*
$$a_d(n) = -b_n(d).$$

Our goal is to provide analogues of this duality for forms that arise in the theory of Borcherds products for Hilbert modular forms. To make this precise, suppose that $p \equiv 1 \pmod 4$ is prime and let χ_p denote the Dirichlet character $\left(\frac{\cdot}{p}\right)$. For an integer $k \geq 0$ and $\epsilon = \pm 1$ let $A_k^{\epsilon}(\Gamma_0(p), \chi_p)$ denote the space of nearly holomorphic modular forms $F(z)$ of weight k and character χ_p such that if $F(z) = \sum c(n)q^n$, then $c(n) = 0$ for $\chi_p(n) = -\epsilon$. Finally, let

$$s(n) = \begin{cases} 2 & n \equiv 0 \pmod p \\ 1 & n \not\equiv 0 \pmod p. \end{cases} \tag{3}$$

In [3], Bruinier and Bundschuh explicitly work out, in terms of classical elliptic modular forms, the Borcherds theory for Hilbert modular forms in the special case of the real quadratic fields $\mathbb{Q}(\sqrt{p})$ for primes $p \equiv 1 \pmod 4$. They show that the coefficients of such an $F(z)$ have an interpretation in terms of infinite product expansions for certain Hilbert modular forms.

Let $K = \mathbb{Q}(\sqrt{p})$ and let $O_K = \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ be the ring of algebraic integers in K. We denote conjugation in K by $x \mapsto \overline{x}$. For $x \in K$, we write $\mathrm{tr}(x) = x + \overline{x}$ and $N(x) = x\overline{x}$

for the trace and norm functions. Let $\mathfrak{d} = (\sqrt{p})$ denote the different of K. For $z \in \mathbb{C}$ we write $z = x + yi$ for $x, y \in \mathbb{R}$. Also, for $z \in \mathbb{C}$, let $e(z) = e^{2\pi i z}$.

The Hilbert modular group $\Gamma_K = SL_2(O_K)$ acts on $\mathbb{H} \times \mathbb{H}$ as follows. If $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ then

$$M \cdot (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{\overline{a}z_2 + \overline{b}}{\overline{c}z_2 + \overline{d}} \right).$$

A Hilbert modular form of weight k is, roughly speaking, a meromorphic function $\Psi(z_1, z_2)$ on $\mathbb{H} \times \mathbb{H}$ such that

$$\Psi(M \cdot (z_1, z_2)) = (cz_1 + d)^k (\overline{c}z_2 + \overline{d})^k \Psi(z_1, z_2).$$

For basic facts about Hilbert modular forms, see [4].

For $m > 0$ define the Hirzebruch-Zagier divisor on the Hilbert modular surface $(\mathbb{H} \times \mathbb{H})/\Gamma_K$ to be the image of

$$T(m) = \bigcup_{\substack{(a,b,\lambda) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{d}^{-1} \\ ab - N(\lambda) = m/p}} \left\{ (z_1, z_2) \in \mathbb{H} \times \mathbb{H} : az_1z_2 + \lambda z_1 + \overline{\lambda} z_2 + b = 0 \right\}. \qquad (4)$$

Here, we understand that all irreducible components of $T(m)$ have multiplicity one. Let

$$S(m) = \bigcup_{\substack{\lambda \in \mathfrak{d}^{-1} \\ -N(\lambda) = m/p}} \{ (z_1, z_2) \in \mathbb{H} \times \mathbb{H} : \lambda y_1 + \overline{\lambda} y_2 = 0 \}. \qquad (5)$$

For $W \subseteq \mathbb{H} \times \mathbb{H}$ and $\lambda \in \mathfrak{d}^{-1}$ we write $(W, \lambda) > 0$ if $\lambda y_1 + \overline{\lambda} y_2 > 0$ for all $(z_1, z_2) \in W$.

Finally, we state Bruinier and Bundschuh's result (Theorem 9 of [3]).

Theorem (Bruinier and Bundschuh)**.** *Let $F = \sum_{n \in \mathbb{Z}} a(n) q^n \in A_0^+(\Gamma_0(p), \chi_p)$ and assume that $s(n)a(n) \in \mathbb{Z}$ for all $n < 0$. Then there is a function $\Psi(z_1, z_2)$ on $\mathbb{H} \times \mathbb{H}$ with the following properties:*

1. *The function Ψ is a meromorphic modular form for Γ_K with some unitary character of finite order. The weight of Ψ is equal to the constant coefficient $a(0)$ of F.*
2. *The divisor of Ψ is determined by the principal part of F. It equals*
$$\sum_{n<0} s(n) a(n) T(-n).$$
3. *Let $W \subset \mathbb{H} \times \mathbb{H}$ be a Weyl chamber attached to F, i.e., a connected component of*
$$\mathbb{H} \times \mathbb{H} - \bigcup_{\substack{n<0 \\ a(n) \neq 0}} S(-n),$$

and let $N = \min\{n : a(n) \neq 0\}$*. The function* Ψ *has the Borcherds product expansion*

$$\Psi(z_1, z_2) = e(\rho_W z_1 + \overline{\rho_W} z_2) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ (\nu, W) > 0}} (1 - e(\nu z_1 + \overline{\nu} z_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}.$$

Here ρ_W *is the Weyl vector associated to* W*, an algebraic number in* K *that can be explicitly computed. The product converges normally for* (z_1, z_2) *with* $y_1 y_2 > |N|/p$ *outside the set of poles.*

Conversely, any meromorphic modular form with a divisor that is a linear combination of Hirzebruch-Zagier divisors has an infinite product expansion of the above form.

For $p = 5, 13$ and 17 and each $m \geq 1$ with $\chi_p(m) \neq -1$, there is a unique $F_{m,p}(z) \in A_0^+(\Gamma_0(p), \chi_p)$ such that

$$F_{m,p}(z) = \frac{1}{s(m)} q^{-m} + O(1) = \sum_{n=-m}^{\infty} A_{m,p}(n) q^n. \tag{6}$$

Similarly, there is a unique

$$G_{m,p}(z) = \frac{1}{s(m)} q^{-m} + O(q) = \sum_{n=-m}^{\infty} B_{m,p}(n) q^n \tag{7}$$

in $A_2^+(\Gamma_0(p), \chi_p)$ (see the appendices for a construction of the $F_{m,p}(z)$ and the $G_{m,p}(z)$). The Zagier duality for the Borcherds exponents is given by the following theorem.

Theorem 1.1. *Suppose that* $p = 5, 13$ *or* 17*. If* m *is a non-negative integer and* $d \geq 1$ *with* $\chi_p(d) \neq -1$ *and* $\chi_p(m) \neq -1$*, then*

$$A_{d,p}(m) = -B_{m,p}(d).$$

Remark. The result stated above only holds for the primes $p = 5$, 13, and 17. Indeed if $p \equiv 1 \pmod 4$, it follows from Theorem 6 of [3] that there is a form

$$F_{1,p}(z) = q^{-1} + O(1) \in A_0^+(\Gamma_0(p), \chi_p)$$

only when there are no weight 2 cusp forms in $A_2^+(\Gamma_0(p), \chi_p)$. A classical result of Hecke implies that the dimension of the space of cusp forms in this space is $\left\lfloor \frac{p-5}{24} \right\rfloor$. Thus, such an $F_{1,p}(z)$ exists only for $p = 5$, 13 and 17.

For larger primes p, the forms in $A_0^+(\Gamma_0(p), \chi_p)$ correspond to Hilbert modular forms whose divisor is a linear combination of Hirzebruch-Zagier divisors (there are no longer any Hilbert modular forms whose divisor is a single Hirzebruch-Zagier divisor). It may be possible to formulate many of these results in this more general context, but there is no longer a natural choice of the $F_{m,p}(z)$ and $G_{m,p}(z)$ (of course, the $G_{m,p}(z)$ are no longer unique).

Remark. The coefficients $B_{m,p}(d)$ are analogous to the coefficients $b_d(m)$, which Zagier interprets as traces of singular moduli. The coefficients $B_{m,p}(d)$ may also be interpreted as traces of singular moduli on the restriction of the corresponding function $\Psi(z_1, z_2)$ to the diagonal $z_1 = z_2$.

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2. Proof of Theorem 1.1

Let $A_k(\Gamma_0(p), \chi_p)$ denote the space of nearly holomorphic forms of weight k and character χ_p. The key idea involved in the proof of Theorem 1.1 is to determine how Hecke operators act on the space $A_0(\Gamma_0(p), \chi_p)$.

Let W_p denote the Fricke involution on $A_k(\Gamma_0(p), \chi_p)$, namely

$$F(z)|W_p = p^{-k/2} z^{-k} F\left(\frac{-1}{pz}\right).$$

It is straightforward to see that if $F(z) \in A_k(\Gamma_0(p), \chi_p)$ then $F(z)|W_p \in A_k(\Gamma_0(p), \chi_p)$, and it is easy to see that $F(z)|W_p|W_p = F(z)$. We define the Fourier expansion of $F(z)$ at zero to be the Fourier expansion of the form $F(z)|W_p$.

Proposition 2.1. *If $F(z) \in A_0(\Gamma_0(p), \chi_p)$, then $F(z)$ is uniquely determined by the principal parts of its Fourier expansions at ∞ and 0.*

Proof. First note that the only cusps of $\Gamma_0(p)$ are ∞ and 0. If $F_1(z), F_2(z) \in A_0(\Gamma_0(p), \chi_p)$ have the same principal part in their Fourier expansions at ∞ and at 0, then $F_3(z) = F_1(z) - F_2(z)$ is holomorphic at each cusp, and since F_1 and F_2 are (by assumption) holomorphic on $\mathbb{H}$, it follows that $F_3(z)$ is a holomorphic modular form of weight 0, which implies that $F_3(z) = c$ is a constant. Suppose that $d \in \mathbb{Z}$ is a quadratic non-residue modulo p. Then, d and p are relatively prime so there exist a and b such that $ad - bp = 1$. Then,

$$F_3\left(\frac{az+b}{pz+d}\right) = \chi_p(d) F_3(z).$$

Since $\chi_p(d) = -1$, this means that $c = -c$ and hence $F_3(z) = c = 0$. Thus, $F_1(z) = F_2(z)$. □

For $F(z) \in A_k^{\epsilon}(\Gamma_0(p), \chi_p)$, there is a relationship between the Fourier expansion at ∞ and at 0. Before we state this relationship, we need some notation. Let $U_p = T_p$ be the usual operator on $A_k(\Gamma_0(p), \chi_p)$ defined as follows. If

$$F(z) = \sum_{n \in \mathbb{Z}} A(n) q^n,$$

then

$$F(z)|U_p = \sum_{n\in\mathbb{Z}} A(pn)q^n. \tag{8}$$

Now, Lemma 3 of [3] gives the relationship between the Fourier expansion at ∞ and 0. Note that our normalization of U_p is slightly different than Bruinier and Bundschuh's.

Lemma 2.2. *Let* $F(z) = \sum_{n\in\mathbb{Z}} A(n)q^n \in A_k(\Gamma_0(p), \chi_p)$ *and* $\epsilon \in \{\pm 1\}$*. Then,* $F(z) \in A_k^{\epsilon}(\Gamma_0(p), \chi_p)$ *if and only if*

$$p^{1-k/2}(F|U_p) = \epsilon\sqrt{p}(F|W_p).$$

Before we describe the action of Hecke operators on the space $A_0(\Gamma_0(p), \chi_p)$, we need some further definitions. For the remainder of this paper we make the following restriction:

$$p = 5, 13, \text{ or } 17.$$

We extend our definitions of the $F_{m,p}(z)$ to include forms in the space $A_0^-(\Gamma_0(p), \chi_p)$. For $\chi_p(m) = -1$, let $F_{m,p}(z)$ be the unique form in $A_0^-(\Gamma_0(p), \chi_p)$ such that

$$F_{m,p}(z) = \sum_{n=-m}^{\infty} A_{m,p}(n)q^n = q^{-m} + O(1). \tag{9}$$

Similarly, for m divisible by p, let $F_{m,p}^-(z)$ be the unique form in $A_0^-(\Gamma_0(p), \chi_p)$ such that

$$F_{m,p}^-(z) = \sum_{n=-m}^{\infty} A_{m,p}^-(n)q^n = \frac{1}{2}q^{-m} + O(1). \tag{10}$$

For a method of constructing the $F_{m,p}(z)$ for $\chi_p(m) = -1$ and $F_{m,p}^-(z)$, see the appendices.

Definition. For $m \geq 1$, let $H_{m,p}(z) \in A_0(\Gamma_0(p), \chi_p)$ be the unique form such that

$$H_{m,p}(z) = O(1)$$

and

$$H_{m,p}(z)|W_p = q^{-m} + O(1).$$

Similarly, for $m \geq 1$ let $I_{m,p}(z) \in A_0(\Gamma_0(p), \chi_p)$ be the unique form such that

$$I_{m,p}(z) = q^{-m} + O(1)$$

and

$$I_{m,p}(z)|W_p = O(1).$$

The following lemmas will provide a description of the action of the Hecke operators on the elements of $A_0(\Gamma_0(p), \chi_p)$. The existence of the $H_{m,p}(z)$ and $I_{m,p}(z)$ is established from these lemmas.

Lemma 2.3. *If* $\gcd(m,p)=1$ *then*

$$\chi_p(m)m(F_{1,p}(z)|T_m) = F_{m,p}(z).$$

Proof. The action of the Hecke operator on $F_{1,p}$ is given by

$$F_{1,p}|T_m = \sum_{n=-\infty}^{\infty}\left(\sum_{d|\gcd(m,n)} \frac{\chi_p(d)}{d} A_{1,p}(mn/d^2)\right) q^n.$$

Now, if $n<0$, then $mn/d^2<0$. Hence, the principal part arises from the terms where $mn/d^2=-1$. Since $d|m$ and $d|n$, it follows that we must have that $m=n=d$. Thus,

$$F_{1,p}|T_m = \frac{\chi_p(m)}{m} q^{-m} + O(1).$$

Thus,

$$\chi_p(m)m(F_{1,p}|T_m) = (\chi_p(m))^2 q^{-m} + O(1) = q^{-m} + O(1).$$

Now, if $\gcd(n,p)=1$ and $\chi_p(n)\neq\chi_p(m)$ then $\chi_p(mn)=-1$ and hence $\chi_p(mn/d^2)=-1$ for all $d|\gcd(m,n)$. Thus, the coefficient of q^n in $F_{1,p}|T_m$ is zero. Thus, $\chi_p(m)m(F_{1,p}|T_m)\in A_0^{\chi_p(m)}(\Gamma_0(p),\chi_p)$. From Lemma 2.2, it follows that such a form is unique and since $F_{m,p}$ satisfies the same conditions we have that

$$\chi_p(m)m(F_{1,p}|T_m) = F_{m,p},$$

as desired. □

Lemma 2.4. *If* $m\geq 1$ *then*

$$H_{m,p}(z) = \frac{1}{\sqrt{p}}\left(F_{pm,p}(z) - F^-_{pm,p}(z)\right).$$

Proof. From Proposition 2.1, it suffices to show that the two forms have the same principal parts in their Fourier expansions at ∞ and at zero. Obviously

$$F_{pm,p}(z) - F^-_{pm,p}(z) = O(1).$$

Also from Lemma 2.2

$$\begin{aligned}
\frac{1}{\sqrt{p}}(F_{pm,p}(z) - F^-_{pm,p}(z))|W_p &= \frac{1}{\sqrt{p}}F_{pm,p}(z)|W_p - \frac{1}{\sqrt{p}}F^-_{pm,p}(z)|W_p \\
&= F_{pm,p}(z)|U_p + F^-_{pm,p}(z)|U_p \\
&= \left(\frac{1}{2}q^{-pm} + O(1)\right)|U_p + \left(\frac{1}{2}q^{-pm} + O(1)\right)|U_p \\
&= q^{-m} + O(1).
\end{aligned}$$

Thus, principal parts of the Fourier expansions of $\frac{1}{\sqrt{p}}(F_{pm,p}(z)-F^{-}_{pm,p}(z))$ and $H_{m,p}(z)$ agree and hence

$$H_{m,p}(z) = \frac{1}{\sqrt{p}}(F_{pm,p}(z) - F^{-}_{pm,p}(z)),$$

as desired. □

Lemma 2.5. *If* $\gcd(m,p) = 1$ *then*

$$H_{m,p}(z) = \chi_p(m)\sqrt{p}(F_{m,p}(z)|U_p).$$

Proof. Note that $F_{m,p}(z)|U_p = (q^{-m} + O(1))|U_p = O(1)$. Thus, $F_{m,p}(z)|U_p$ is holomorphic at ∞. However by Lemma 2.2,

$$\begin{aligned}\chi_p(m)\sqrt{p}(F_{m,p}(z)|U_p)|W_p &= \chi_p(m)\sqrt{p}\left(\frac{\chi_p(m)}{\sqrt{p}}F_{m,p}(z)\right)|W_p|W_p \\ &= F_{m,p}(z) = q^{-m} + O(1).\end{aligned}$$

Thus, the two functions have the same principal parts of the Fourier expansions at ∞ and zero and hence they agree. □

Lemma 2.6. *If* $p|m$ *then*

$$H_{pm,p}(z) = p(H_{m,p}(z)|U_p).$$

Proof. Since $H_{m,p}(z)$ is holomorphic at ∞ so is $H_{m,p}(z)|U_p$. Now, from Lemma 2.4,

$$\begin{aligned}p(H_{m,p}(z)|U_p)|W_p &= p\left(\frac{1}{\sqrt{p}}F_{pm,p}(z) - \frac{1}{\sqrt{p}}F^{-}_{pm,p}(z)\right)|U_p|W_p \\ &= \left(F_{pm,p} + F^{-}_{pm,p}\right)|W_p|W_p \\ &= F_{pm,p} + F^{-}_{pm,p} = q^{-pm} + O(1).\end{aligned}$$

The desired result follows since the principal parts of the two functions are the same at both cusps. □

Lemma 2.7. *For* $m \geq 1$,

$$I_{m,p}(z) = (F_{pm,p}(z) + F^{-}_{pm,p}(z))|U_p.$$

Proof. Clearly,

$$\begin{aligned}(F_{pm,p} + F^{-}_{pm,p})|U_p &= \left(\frac{1}{2}q^{-pm} + O(1)\right)|U_p + \left(\frac{1}{2}q^{-pm} + O(1)\right)|U_p \\ &= 2\left(\frac{1}{2}q^{-m} + O(1)\right) = q^{-m} + O(1).\end{aligned}$$

Also,

$$\begin{aligned}(F_{pm,p} + F^-_{pm,p})|U_p|W_p &= \frac{1}{\sqrt{p}} F_{pm,p}|W_p|W_p - \frac{1}{\sqrt{p}} F^-_{pm,p}|W_p|W_p \\ &= \frac{1}{\sqrt{p}} F_{pm,p} - \frac{1}{\sqrt{p}} F^-_{pm,p} = O(1).\end{aligned}$$

Thus, the functions have the same Fourier expansion at each cusp, so they are equal. □

Lemma 2.8. *For $m \geq 1$,*

$$I_{m,p}(z) = \begin{cases} F_{m,p}(z) & \text{if } \gcd(m,p) = 1, \\ F_{m,p}(z) + F^-_{m,p}(z) & \text{if } p|m. \end{cases}$$

Proof. Note that if $\gcd(m,p) = 1$ then the Fourier expansions of $F_{m,p}$ and $I_{m,p}$ at infinity agree. The Fourier expansion of $F_{m,p}$ at zero is

$$\begin{aligned}F_{m,p}|W_p &= \frac{\chi_p(m)}{\sqrt{p}} F_{m,p}|U_p \\ &= \frac{\chi_p(m)}{\sqrt{p}} (q^{-m} + O(1))|U_p = O(1).\end{aligned}$$

Thus, $F_{m,p}$ and $I_{m,p}$ have the same Fourier expansions at each cusp provided m and p are coprime, and hence for such m, $F_{m,p} = I_{m,p}$.

If $p|m$, then

$$F_{m,p} + F^-_{m,p} = \left(\frac{1}{2} q^{-m} + O(1)\right) + \left(\frac{1}{2} q^{-m} + O(1)\right) = q^{-m} + O(1).$$

At zero,

$$\begin{aligned}\left(F_{m,p} + F^-_{m,p}\right)|W_p &= F_{m,p}|W_p + F^-_{m,p}|W_p \\ &= \sqrt{p} F_{m,p}|U_p - \sqrt{p} F^-_{m,p}|U_p \\ &= (\sqrt{p} q^{-m} + O(1))|U_p + (-\sqrt{p} q^{-m} + O(1))|U_p = O(1).\end{aligned}$$

Thus, $F_{m,p} + F^-_{m,p}$ and $I_{m,p}$ have the same Fourier expansion at each cusp and hence $I_{m,p} = F_{m,p} + F^-_{m,p}$, as desired. □

Remark. The above lemmas give an expression for the coefficients of $F_{m,p}(z)$ in terms of the coefficients of $F_{1,p}(z)$.

We will now use the lemmas to work towards a proof of Theorem 1.1.

Lemma 2.9. *Suppose that m and n are positive integers coprime to p. Then,*

$$\chi_p(n) n A_{m,p}(n) = \chi_p(m) m A_{n,p}(m).$$

Proof. From Lemma 2.3, we know that

$$F_{m,p} = \chi_p(m)m(F_{1,p}|T_m)$$
$$F_{n,p} = \chi_p(n)n(F_{1,p}|T_n).$$

Now, the nth coefficient of $(\chi_p(n)n)F_{m,p}$ is

$$\chi_p(n)nA_{m,p}(n) = mn\chi_p(mn) \sum_{d|\gcd(m,n)} \frac{\chi_p(d)}{d} A_{1,p}(mn/d^2).$$

Similarly, the mth coefficient of $(\chi_p(m)m)F_{n,p}$ is

$$\chi_p(m)mA_{n,p}(m) = mn\chi_p(mn) \sum_{d|\gcd(m,n)} \frac{\chi_p(d)}{d} A_{1,p}(nm/d^2).$$

Thus,

$$\chi_p(n)nA_{m,p}(n) = \chi_p(m)mA_{n,p}(m),$$

as desired. □

Lemma 2.10. *Suppose that* $p|m$ *and* $\gcd(n,p) = 1$*. Then,*

$$\chi_p(n)n\left[A_{m,p}(n) - A^-_{m,p}(n)\right] = mA_{n,p}(m).$$

Proof. Write $m = p^b m'$ with m' coprime to p. From Lemma 2.3,

$$\chi_p(m')m'(F_{1,p}|T_{m'}) = F_{m',p}.$$

From Lemma 2.5,

$$\chi_p(m')\sqrt{p}F_{m',p}|U_p = H_{m',p}.$$

From a repeated application of Lemma 2.6,

$$p^{b-1}H_{m',p}|U_p^{b-1} = H_{p^{b-1}m',p}.$$

Finally, from Lemma 2.4,

$$H_{p^{b-1}m',p} = \frac{1}{\sqrt{p}}(F_{p^b m'} - F^-_{p^b m'}).$$

Putting all these equations together gives that

$$\begin{aligned} F_{p^b m',p} - F^-_{p^b m',p} &= \sqrt{p}H_{p^{b-1}m',p} = \sqrt{p}p^{b-1}H_{m',p}|U_p^{b-1} \\ &= \chi_p(m')p^b F_{m',p}|U_p^b = p^b F_{m',p}|U_p^b \\ &= p^b m' F_{1,p}|T_{p^b m'}. \end{aligned}$$

Writing $m = p^b m'$, multiplying by $\chi_p(n)n$ and considering the nth coefficient gives

$$\chi_p(n)nA_{m,p}(n) - \chi_p(n)nA^-_{m,p}(n) = m\chi_p(n)n \sum_{d|\gcd(m,n)} \chi_p(d)d^{-1}A_{1,p}(mn/d^2). \quad (11)$$

Lemma 2.3 also implies that

$$F_{n,p} = \chi_p(n)n(F_{1,p}|T_n).$$

Considering the mth coefficient and multiplying by m gives

$$mA_{n,p}(m) = m\chi_p(n)n \sum_{d|\gcd(m,n)} \chi_p(d)d^{-1}A_{1,p}(mn/d^2). \tag{12}$$

Now, the right hand sides of equations (11) and (12) agree and hence the left hand sides agree. This gives

$$mA_{n,p}(m) = \chi_p(n)nA_{m,p}(n) - \chi_p(n)nA^-_{m,p}(n),$$

as desired. □

The previous lemmas above imply the following lemma, which is the key to the proof of Theorem 1.1.

Lemma 2.11. *If m and n are positive integers with $\chi_p(m) \neq -1$ and $\chi_p(n) \neq -1$, then $mA_{n,p}(m) = nA_{m,p}(n)$.*

Proof. From Lemma 2.9 and Lemma 2.10, it suffices to prove the result when $p|m$ and $p|n$. Write $m = p^a m'$ and $n = p^b n'$ with m' and n' coprime to p. Assume without loss of generality that $a \geq b$.

First, suppose that $a > b$. From Lemmas 2.4 and 2.6

$$F_{m,p} - F^-_{m,p} = \sqrt{p}H_{m/p,p} = p^b\sqrt{p}H_{m/p^{b+1},p}|U_p^b = p^b(F_{m/p^b,p} - F^-_{m/p^b,p})|U_p^b. \tag{13}$$

From Lemmas 2.8 and 2.7, it follows that

$$(F_{m,p} + F^-_{m,p})|U_p^b = I_{m/p^b,p} = F_{m/p^b,p} + F^-_{m/p^b,p}. \tag{14}$$

Taking the nth coefficient on both sides of equation (13) and the n/p^bth coefficient on both sides of equation (14) gives

$$\begin{aligned} A_{m,p}(n) - A^-_{m,p}(n) &= p^b A_{m/p^b,p}(p^b n) - p^b A^-_{m/p^b,p}(p^b n) \\ A_{m,p}(n) + A^-_{m,p}(n) &= A_{m/p^b,p}(n/p^b) + A^-_{m/p^b,p}(n/p^b). \end{aligned}$$

Adding and multiplying by n gives

$$2nA_{m,p}(n) = p^b nA_{m/p^b,p}(p^b n) - p^b nA^-_{m/p^b,p}(p^b n) + nA_{m/p^b,p}(n/p^b) + nA^-_{m/p^b,p}(n/p^b). \tag{15}$$

Now, Lemmas 2.4, 2.5, and 2.6 imply that

$$\begin{aligned} F_{n,p} - F^-_{n,p} = \sqrt{p}H_{n/p,p} &= \sqrt{p}p^{b-1}H_{n/p^b,p}|U_p^{b-1} \\ &= \sqrt{p}p^{b-1}\chi_p(n')\sqrt{p}(F_{n',p}|U_p)|U_p^{b-1} \\ &= \chi_p(n')p^b F_{n',p}|U_p^b. \end{aligned}$$

Taking the mth coefficient of this equation yields

$$A_{n,p}(m) - A^{-}_{n,p}(m) = \chi_p(n')p^b A_{n',p}(p^b m). \tag{16}$$

Now, Lemmas 2.7 and 2.8 imply that

$$(F_{n,p} + F^{-}_{n,p})|U_p^b = I_{n',p} = F_{n',p}.$$

Taking the m/p^bth coefficient of this equation yields

$$A_{n,p}(m) + A^{-}_{n,p}(m) = A_{n',p}(m/p^b). \tag{17}$$

Now, adding (16) and (17) and multiplying by m gives

$$2mA_{n,p}(m) = m\chi_p(n')p^b A_{n',p}(p^b m) + mA_{n',p}(m/p^b). \tag{18}$$

Now, I will show that the right hand side of (15) is equal to the right hand side of (18).

By Lemma 2.10, we have that

$$mA_{n',p}(m/p^b) = p^b(m/p^b)A_{n',p}(m/p^b) = p^b\chi_p(n')n'\left[A_{m/p^b,p}(n') - A^{-}_{m/p^b,p}(n')\right].$$

Note that

$$p^b\chi_p(n')n'\left[A_{m/p^b,p}(n') - A^{-}_{m/p^b,p}(n')\right] = n\left[A_{m/p^b,p}(n') + A^{-}_{m/p^b,p}(n')\right]$$

since $A_{m/p^b,p}(n') = 0$ if $\chi_p(n') = -1$ and $A^{-}_{m/p^b,p}(n') = 0$ if $\chi_p(n') = 1$. Thus, the second term on the right hand side of (18) is equal to the sum of the third and fourth terms on the right hand side of (15).

Again, by Lemma 2.10, we have that

$$m\chi_p(n')p^b A_{n',p}(p^b m) = n'\chi_p(n')\left(\chi_p(n')\left[A_{p^b m,p}(n') - A^{-}_{p^b m,p}(n')\right]\right). \tag{19}$$

Now, from Lemmas 2.4 and 2.6

$$\begin{aligned} F_{p^b m,p} - F^{-}_{p^b m,p} &= \sqrt{p}H_{p^{b-1}m,p} = \sqrt{p}p^{2b}H_{m/p^{b+1},p}|U_p^{2b} \\ &= p^{2b}\left(F_{m/p^b,p} - F^{-}_{m/p^b,p}\right)|U_p^{2b}. \end{aligned}$$

Taking the n'th coefficient gives

$$A_{p^b m,p}(n') - A^{-}_{p^b m,p}(n') = p^{2b}\left[A_{m/p^b,p}(p^b n) - A^{-}_{m/p^b,p}(p^b n)\right].$$

Plugging this into (19) gives

$$m\chi_p(n')p^b A_{n',p}(p^b m) = np^b\left[A_{m/p^b,p}(np^b) - A^{-}_{m/p^b,p}(np^b)\right].$$

Hence, the first term on the right hand side of (18) equals the sum of the first two terms on the right hand side of (15). Hence, the right hand side of (18) equals the right hand side of (15) giving

$$nA_{m,p}(n) = mA_{n,p}(m),$$

as desired.

Second, suppose that $a = b$. This case is analogous to when $a > b$, but with simpler identities. From Lemmas 2.4, 2.6, and 2.5,

$$\begin{aligned} F_{m,p} - F^-_{m,p} &= \sqrt{p} H_{m/p,p} = \sqrt{p} p^{b-1} H_{m/p^b,p} | U_p^{b-1} \\ &= \sqrt{p} p^{b-1} \left(\chi_p(m/p^b) \sqrt{p} F_{m/p^b,p} | U_p \right) | U_p^{b-1} \\ &= p^b \chi_p(m/p^b) F_{m/p^b,p} | U_p^b. \end{aligned}$$

Taking the nth coefficient of both sides gives

$$A_{m,p}(n) - A^-_{m,p}(n) = p^b \chi_p(m') A_{m',p}(p^b n). \tag{20}$$

Also, Lemmas 2.8 and 2.7 imply that

$$\begin{aligned} F_{m',p} &= I_{m',p} = I_{p^{b-1}m',p} | U_p^{b-1} \\ &= \left(F_{m,p} + F^-_{m,p} \right) | U_p^b. \end{aligned}$$

Taking the n'th coefficient of both sides gives

$$A_{m,p}(n) + A^-_{m,p}(n) = A_{m',p}(n').$$

Adding this equation to equation (20) and multiplying by n gives

$$2nA_{m,p}(n) = np^b \chi_p(m') A_{m',p}(p^b n) + nA_{m',p}(n'). \tag{21}$$

Since the largest power of p dividing m and n is the same, analogous arguments show that the same equation is true with n in place of m. Specifically,

$$2mA_{n,p}(m) = mp^b \chi_p(n') A_{n',p}(p^b m) + mA_{n',p}(m'). \tag{22}$$

To complete the proof, it suffices to show that the right hand sides of (21) and (22) agree.

First, note that if $\chi_p(n') \neq \chi_p(m')$ then $A_{n',p}(m') = A_{m',p}(n') = 0$. On the other hand, if $\chi_p(n') = \chi_p(m')$ then $\chi_p(m'n') = 1$. Lemma 2.9 implies in this case that

$$\begin{aligned} nA_{m',p}(n') &= p^b \chi_p(m') \chi_p(n') n' A_{m',p}(n') \\ &= p^b \chi_p(m') \chi_p(m') m' A_{n',p}(m') \\ &= mA_{n',p}(m'), \end{aligned}$$

as desired. Thus, the second term on the right hand side of equation (21) equals the second term on the right hand side of equation (22).

Second, from Lemma 2.10, it follows that

$$np^b \chi_p(m') A_{m',p}(p^b n) = m' \left[A_{p^b n,p}(m') - A^-_{p^b n}(m') \right].$$

Now, from Lemmas 2.4, 2.5, and 2.6, it follows that

$$\begin{aligned} F_{p^b n,p} - F^-_{p^b n,p} &= \sqrt{p} H_{p^b n,p} = \sqrt{p} p^{2b-1} H_{n/p^b,p} | U_p^{2b-1} \\ &= p^{2b} \chi_p(n') F_{n',p} | U_p^{2b}. \end{aligned}$$

Hence,

$$m' \left[A_{p^b n,p}(m') - A^{-}_{p^b n,p}(m') \right] = m' p^{2b} \chi_p(n') F_{n',p}(p^b m) = m p^b \chi_p(n') A_{n',p}(p^b m).$$

Thus, combining all of the above equations, we have

$$n p^b \chi_p(m') A_{m',p}(p^b n) = m p^b \chi_p(n') A_{n',p}(p^b m),$$

and hence the first term on the right hand side of equation (21) equals the first term on the right hand side of equation (22). Thus, the left hand sides of equation (21) and (22) are equal, giving $nA_{m,p}(n) = mA_{n,p}(m)$, as desired. □

Proof of Theorem 1.1. Note that if $m = 0$, then the result follows from Theorem 6 of [3]. Assume therefore that $m > 0$.

Note that

$$\frac{d}{dz} F\left(\frac{az+b}{cz+d}\right) = F'\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2}.$$

Hence, if $F(z) \in A_0(\Gamma_0(p), \chi_p)$, then $F'(z)$ transforms like a weight two modular form. Also, since the Fourier expansion $F(z) = \sum_{n=-h}^{\infty} c(n) q^n$ converges normally in $\mathbb{H}$,

$$F'(z) = \sum_{n=-h}^{\infty} (2\pi i n) c(n) q^n. \tag{23}$$

Thus, $F'(z)$ is meromorphic at ∞. Similarly, differentiating $F(z)|W_p$ shows that $F'(z)|W_p$ is meromorphic at ∞ and hence $F'(z) \in A_2(\Gamma_0(p), \chi_p)$.

Now, from (23), we see that if $\chi_p(d) \neq -1$ then $F'_{d,p}(z) \in A_2^{+}(\Gamma_0(p), \chi_p)$. Moreover,

$$-\frac{1}{2\pi i d} F'_{d,p}(z) = G_{d,p}(z).$$

Hence,

$$A_{d,p}(m) = -\frac{d}{m} B_{d,p}(m).$$

Thus, from Lemma 2.11

$$\begin{aligned} mA_{d,p}(m) &= dA_{m,p}(d) \\ &= d\left(\frac{-m}{d}\right) B_{m,p}(d) \\ &= -mB_{m,p}(d). \end{aligned}$$

Dividing by m gives $A_{d,p}(m) = -B_{m,p}(d)$, as desired. □

3. Appendix 1. Data for $p = 5$

Let

$$K_{2,5}(z) = 1 + \frac{2}{L(-1,\chi_5)} \sum_{n=1}^{\infty} \sum_{d|n} d\chi_5(d) q^n = \eta(z)^5/\eta(5z)$$

$$L_{2,5}(z) = \sum_{n=1}^{\infty} \sum_{d|n} d\chi_5(n/d) q^n = \eta(5z)^5/\eta(z)$$

be the two weight 2 Eisenstein series for $M_2(\Gamma_0(5),\chi_5)$, where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ is the usual Dedekind η-function. Here, $L(s,\chi_p)$ denotes the analytic continuation of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\chi_p(n)}{n^s}.$$

Note that $L(-1,\chi_5) = -2/5$. Let

$$M_{2,5}(z) = 1 + 6\sum_{n=1}^{\infty}(\sigma(n) - 5\sigma(n/5))q^n \in M_2(\Gamma_0(5))$$

be the usual weight 2 Eisenstein series for $\Gamma_0(5)$ (no character). Then, the following are formulas for $F_{1,5}(z), \ldots, F_{5,5}(z), F^-_{5,5}(z)$.

$$\begin{aligned}
F_{1,5}(z) &= \frac{M_{2,5}(z)}{L_{2,5}(z)} \\
F_{2,5}(z) &= F_{1,5}(z)\frac{K_{2,5}(z)}{L_{2,5}(z)} + F_{1,5}(z) \\
F_{3,5}(z) &= F_{1,5}(z)^3 - 15F_{2,5}(z) - 108F_{1,5}(z) \\
F_{4,5}(z) &= F_{2,5}(z)F_{1,5}(z)^2 - 10F_{3,5}(z) - 42F_{2,5}(z) - 60F_{1,5}(z) \\
F_{5,5}(z) &= \frac{(H_{2,5}(z) - 5I_{2,5}(z))E_4(5z)E_6(5z)}{2\Delta(5z)} + 5F_{4,5}(z) + 15F_{1,5}(z) \\
F^-_{5,5}(z) &= \frac{(H_{2,5}(z) + 5I_{2,5}(z))E_4(5z)E_6(5z)}{2\Delta(5z)} - 5F_{3,5}(z) - 10F_{2,5}(z).
\end{aligned}$$

In the last equation, $E_4(z)$, $E_6(z)$ are the usual weight 4 and 6 Eisenstein series, respectively, and $\Delta(z) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$ is the usual cusp form of weight 12. Note that the linear combinations $H_{2,5}(z) \mp 5I_{2,5}(z)$ is chosen so that $H_{2,5}(z) \mp 5I_{2,5}(z) \in A_2^{\pm}(\Gamma_0(5),\chi_5)$.

The other $F_{m,5}(z)$ can be computed inductively by multiplying $F_{m-5,5}(z)$ by $j(5z) = E_4(5z)^3/\Delta(5z)$ and subtracting off linear combinations of $F_{m',5}(z)$ for $m' < m$. The following is a table of q-expansions for $F_{m,5}(z)$ for small m.

$$
\begin{aligned}
F_{1,5}(z) &= q^{-1} + 5 + 11q - 54q^4 + 55q^5 + 44q^6 + \cdots \\
F_{2,5}(z) &= q^{-2} - 5 + 119q^2 - 88q^3 - 680q^5 + 3672q^7 + \cdots \\
F_{3,5}(z) &= q^{-3} - 10 - 132q^2 + 1196q^3 - 4485q^5 - 13032q^7 + \cdots \\
F_{4,5}(z) &= q^{-4} + 15 - 216q + 4959q^4 + 22040q^5 - 90984q^6 + \cdots \\
F_{5,5}(z) &= \frac{1}{2}q^{-5} + 15 + 275q + 27550q^4 + 43893q^5 + 255300q^6 + \cdots \\
F^{-}_{5,5}(z) &= \frac{1}{2}q^{-5} - 10 - 1700q^2 - 7475q^3 - 43882q^5 - 685950q^7 + \cdots
\end{aligned}
$$

The following are formulas for $G_{0,5}(z), \ldots, G_{4,5}(z)$ (here we define $G_{m,p}(z) \in A_2^{\chi_p(m)}(\Gamma_0(p), \chi_p)$ in an analogous way to the $F_{m,p}(z)$). The other $G_{m,5}(z)$ can be computed inductively in the same way as the $F_{m,5}(z)$.

$$
\begin{aligned}
G_{0,5}(z) &= \frac{K_{2,5}(z) - 5L_{2,5}(z)}{2} \\
G^{-}_{0,5}(z) &= \frac{K_{2,5}(z) + 5L_{2,5}(z)}{2} \\
G_{1,5}(z) &= \frac{K_{2,5}(z)^2}{L_{2,5}(z)} + 11K_{2,5}(z) \\
G_{2,5}(z) &= 2G_{0,5}(z)F_{1,5}(z)^2 - 72G_{0,5}(z) + 178G^{-}_{0,5}(z) \\
G_{3,5}(z) &= G_{1,5}(z)F_{1,5}(z)^2 - 10G_{2,5}(z) - 36G_{1,5}(z) \\
G_{4,5}(z) &= G_{2,5}(z)F_{1,5}(z)^2 - 10G_{3,5}(z) - 47G_{2,5}(z) - 110G_{1,5}(z) - 2G_{0,5}(z) - 2G^{-}_{0,5}(z).
\end{aligned}
$$

The following is a table of the q-expansions for $G_{m,5}(z)$ for small m.

$$
\begin{aligned}
G_{0,5}(z) &= \frac{1}{2} - 5q - 15q^4 - 15q^5 - 10q^6 + \cdots \\
G_{0,5}(z)^{-} &= \frac{1}{2} + 5q^2 + 10q^3 + 10q^5 + 30q^7 + \cdots \\
G_{1,5}(z) &= q^{-1} - 11q + 216q^4 - 275q^5 - 264q^6 + \cdots \\
G_{2,5}(z) &= q^{-2} - 119q^2 + 132q^3 + 1700q^5 - 12852q^7 + \cdots \\
G_{3,5}(z) &= q^{-3} + 88q^2 - 1196q^3 + 7475q^5 + 30408q^7 + \cdots \\
G_{4,5}(z) &= q^{-4} + 54q - 4959q^4 - 27550q^5 + 136476q^6 + \cdots .
\end{aligned}
$$

4. Appendix 2. Tables for $p = 13$

As with $p = 5$, let

$$K_{2,13}(z) = 1 + \frac{2}{L(-1,\chi_{13})}\sum_{n=1}^{\infty}\sum_{d|n} d\chi_{13}(d)q^n, \quad L_{2,13}(z) = \sum_{n=1}^{\infty}\sum_{d|n} d\chi_{13}(n/d)q^n$$

be the weight 2 Eisenstein series for $M_2(\Gamma_0(13),\chi_{13})$. Here $L(-1,\chi_{13}) = -2$. Let

$$M_{2,13}(z) = 1 + 2\sum_{n=1}^{\infty}(\sigma(n) - 13\sigma(n/13))q^n \in M_2(\Gamma_0(13))$$

be the usual weight 2 Eisenstein series. The $F_{1,13}(z)$ and $F_{2,13}(z)$ are given by

$$F_{1,13}(z) = \frac{M_{2,13}(z)}{L_{2,13}(z)}$$

$$F_{2,13}(z) = \frac{K_{2,13}(z)}{L_{2,13}(z)}F_{1,13}(z) + F_{1,13}(z).$$

Formulas for $F_{m,13}(z)$ for $m \geq 3$ not a multiple of 13 can be found by taking $F_{m-2,13}(z)F_{1,13}(z)^2$ and subtracting off multiples of $F_{m',13}(z)$ for $m' < m$. This is similar to the formulas above for $F_{3,5}(z)$ and $F_{4,5}(z)$. Formulas for $F_{13,13}(z)$ and $F^{-}_{13,13}(z)$ can be found similar to above. Finally, as above, one can multiply $F_{m,13}(z)$ by $j(13z)$ and subtract a linear combination of $F_{m',13}(z)$ for $m' < m + 13$ to get $F_{m+13,13}(z)$. By arguing as in the $p = 5$ case, one can easily construct the $G_{m,13}(z)$ in terms of the above modular forms.

To prove the formulas above, note that Theorem 6 of [3] guarantees the existence of $F_{m,13}(z)$ and that Proposition 2.1 and Lemma 2.2 guarantee the uniqueness. Therefore $F_{m,13}(z)L_{2,13}(z)^m$ must be holomorphic, and comparing coefficients gives the above formulas.

5. Appendix 3. Tables for $p = 17$

Let

$$K_{2,17}(z) = 1 + \frac{2}{L(-1,\chi_{17})}\sum_{n=1}^{\infty}\sum_{d|n} d\chi_{17}(d)q^n, \quad L_{2,17}(z) = \sum_{n=1}^{\infty}\sum_{d|n} d\chi_{17}(n/d)q^n$$

be the weight 2 Eisenstein series for $M_2(\Gamma_0(17),\chi_{17})$. Here, $L(-1,\chi_{17}) = -4$. Let

$$M_{2,17}(z) = 1 + \frac{3}{2}\sum_{n=1}^{\infty}(\sigma(n) - 17\sigma(n/17))q^n \in M_2(\Gamma_0(17))$$

be the usual weight 2 Eisenstein series. Let

$$S_{2,17}(z) = q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + \cdots$$

be the weight 2 cusp form in $S_2(\Gamma_0(17))$ associated to the elliptic curve $X_0(17)$. Then, the following are formulas for $F_{1,17}(z)$ and $F_{2,17}(z)$. They are more complicated because the modular curve $X_0(17)$ has genus 1 (while $X_0(5)$ and $X_0(13)$ have genus 0).

$$F_{1,17}(z) = \frac{K_{2,17}(z)}{S_{2,17}(z)}$$
$$F_{2,17}(z) = \frac{4M_{2,17}(z)^2 - 4M_{2,17}(z)S_{2,17}(z) - 24S_{2,17}(z)^2 - 17L_{2,17}(z)^2}{4S_{2,17}(z)L_{2,17}(z)}.$$

Formulas for the other $F_{m,17}(z)$ can be determined as in the cases $p = 5$ and $p = 13$. Again, one can easily construct the $G_{m,17}(z)$ in terms of the above modular forms.

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