Math 112
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Sample Homework Solutions

1) Find the exact value of the integral using a Riemann sum:

$\int_{-1}^{2} (1+x^2)\,dx$:

$b=2$
$a=-1$
$b-a=3$

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

We need to set up an Riemann sum with n equal subintervals and a right-hand method. So,

$$x_i = a + \frac{b-a}{n} i = -1 + \frac{3i}{n}$$

The height of the i^{th} rectangle is $f(x_i) = 1 + x_i^2$

$$= 1 + \left(-1 + \frac{3i}{n}\right)^2$$

$$= 1 + 1 - \frac{6i}{n} + \frac{9i^2}{n^2}$$

$$= 2 - \frac{6i}{n} + \frac{9i^2}{n^2}$$

So, the area of the i^{th} rectangle is $bh = f(x_i)\Delta x$

$$= \left(2 - \frac{6i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n}$$

$$= \frac{6}{n} - \frac{18i}{n^2} + \frac{27i^2}{n^3}$$

The total area with n rectangles is:

$$A_n = \sum_{i=1}^{n}\left(\frac{6}{n} - \frac{18i}{n^2} + \frac{27i^2}{n^3}\right) = \frac{6}{n}\sum_{i=1}^{n} 1 - \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{6}{\cancel{n}}\cancel{n} - \frac{\cancel{18}^{9}}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\cancel{27}^{9}}{n^{\cancel{3}2}}\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2}$$

$$= 6 - 9\left(\frac{n+1}{n}\right) + \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}$$

Finally, the exact area is $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(6 - 9\frac{n+1}{n} + \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right)$

$$= 6 - 9 + \frac{9}{2}\cdot 2 = \boxed{6}$$

because $\lim_{n\to\infty} \frac{n+1}{n} = 1$ (both top powers are 1 with coefficient 1)

and $\lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = 2$ (both top powers are 2 with coefficients $\frac{2}{1} = 2$).

2) Determine a region whose area is equal to the given limit.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(1 + \frac{4i}{n}\right) \cdot \frac{4}{n}$$

First, we find that $\Delta x = \frac{4}{n}$, so $b - a = 4$.

Next, we notice that $f(x_i) = \sin\left(1 + \frac{4i}{n}\right)$, so

$f(x) = \sin(x)$ and $x_i = 1 + \frac{4i}{n}$, so $a = 1$.

Thus $b = a + 4 = 1 + 4 = 5$.

Hence this limit represents the area under the function $f(x) = \sin x$ from $x = 1$ to $x = 5$; or $\int_1^5 \sin(x)\,dx$.