# NEUMANN FIXED BOUNDARY REGULARITY FOR AN ELLIPTIC FREE BOUNDARY PROBLEM

S. RAYNOR

ABSTRACT. We examine the regularity properties of solutions to an elliptic free boundary problem near a Neumann fixed boundary. Consider a nonnegative function $u$, defined variationally, which is harmonic where it is positive and satisfies a gradient jump condition weakly along the free boundary $\partial\{u > 0\}$. Our main result is that $u$ is Lipschitz continuous. Additionally, we prove various basic properties of such a minimizer near a portion of the fixed boundary on which $\frac{\partial u}{\partial \nu} = 0$ weakly. Our results include up-to-the boundary gradient estimates on harmonic functions with Neumann boundary conditions on convex domains, which have independent interest.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $Q$ be a smooth, bounded, nonnegative function on $\Omega$. The subject of this paper will be a function $u$ with the following properties:

$$
\begin{aligned}
u(x) &\geq 0 && \forall\ x \in \Omega \\
\Delta u(x) &= 0 && \forall\ x \in \{u > 0\} \cap \Omega \\
|\nabla u(x)| &= Q(x) && \forall\ x \in \partial\{u > 0\} \cap \Omega
\end{aligned} \tag{1}
$$

The free boundary is the set $\Lambda = \partial\{u > 0\} \cap \Omega$. This free boundary problem has several applications, particularly to the study of jets and cavities. It is also an excellent model problem for the many free boundary problems that occur naturally in science and industry. See, for example, the other papers on the subject by Alt, Caffarelli, and Friedman (particularly [2] and its references), as well as the book on free boundary problems by Friedman.[7]

The one-phase free boundary problem treated here was first considered by Alt and Caffarelli in [1]. They first concluded that a minimizer exists for the functional $J$ with nonnegative Dirichlet boundary conditions, then computed the basic properties of local minima, including that they are globally subharmonic and harmonic in their positive phase. They also proved that the free boundary condition holds in a weak sense. Alt and Caffarelli proceeded to examine the question of interior regularity, and concluded that a local minimum $u$ is locally Lipschitz continuous. They found that, with $0 < m \leq Q$ and $Q$ smooth, the free boundary is a $C^{1,\alpha}$ surface except at a set of 0 surface measure. Moreover, for $n = 2$ the free boundary is analytic if $Q$ is. They finally produced a non-minimizing global solution in dimension 3 with a point singularity in the free boundary at the origin, and several other examples. Fixed boundary regularity of $u$ has not been approached using variational techniques. However, uniform regularization methods have been used to study the regularity of solutions near a smooth boundary. In this technique, the free boundary problem is modeled by a smooth semilinear PDE. This method has the advantage of applying to a larger class of solutions, but cannot be used if the regularity of $\partial\Omega$ is too low, as is the case in the current work. In [4], Berestycki, Caffarelli, and Nirenberg used these methods to establish uniform Lipschitz continuity up to a smooth Neumann boundary. They establish that these solutions do indeed converge to a Lipschitz continuous weak solution to the free boundary problem, and obtain some control over the shape of the free

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boundary as well. In [8], Gurevich examined the uniform regularity of this singular perturbation of the problem, near a boundary with smooth non-trivial dirichlet data $u_0$. He concluded that an extra condition ($|\nabla u_0| = 0$ when $u_0 = 0$) is necessary and sufficient to obtain uniform Lipschitz continuity of the regularized solutions $u_\epsilon$, which implies Lipschitz continuity of the uniform limit $u$. This condition is automatically satisfied in the one-phase problem if $u_0$ is smooth, so Lipschitz continuity does hold in that case.

In this work, we will study fixed boundary regularity for this problem near a Neumann boundary using variational techniques. This will allow us to place minimal regularity restrictions on the domain $\Omega$. The variational approach taken here allows one to consider a $u$ which is a priori not smooth enough for the free boundary condition in (1) to make sense pointwise. The free boundary condition can be recovered almost everywhere after some regularity results are obtained, if $Q$ is strictly positive and smooth [1]. This approach also enables one to obtain existence and basic properties of $u$ fairly easily. The cost is that we consider only (local) minimizers to a related functional and not general solutions to the problem described above. This is a nontrivial restriction, as it is known (see above) that nonminimizing solutions sometimes have strictly lower regularity. Our main result is:

**Theorem 1.** *Suppose that $\partial\Omega$ is convex. There exists a constant $C$ depending only on the dimension, the Lipschitz character of $\partial\Omega$ and $Q(x)$ such that if $u$ solves (1) in a variational sense, then for almost every $x \in \Omega$, $|\nabla u(x)| \le C$.*

For a more precise statement of the result, please see Theorem 2 in Section 4.

**Remark** $\Omega$ must be convex. Without this condition the theorem fails. Indeed, even harmonic functions in non-convex Lipschitz domains fail to be Lipschitz continuous. However, if $\Omega$ is non-convex one can still find a minimizer $u$, and that minimizer is Hölder continuous for some $0 < \alpha < 1$ determined solely by the Lipschitz character of the fixed boundary, as will be proved in Section 2. If $\partial\Omega$ is smooth, then $\Omega$ can be locally transformed to a convex domain. This transformation will change the laplacian to a smoothly elliptic operator, and the theorem will still hold as stated, with the constant possibly depending on the $C^2$ norm of $\partial\Omega$. Therefore, as the proof is local in nature, the convexity restriction essentially only prevents us from looking at boundary regions that are simultaneously nonsmooth and nonconvex.

The structure of this paper is as follows. Section 2 provides the setup of the paper and basic properties of the solution $u$, including global Hölder continuity. Section 3 gives some necessary properties of positive harmonic functions with Neumann boundary conditions. It includes up-to-the-boundary gradient control lemmas, which will be used significantly in the proof of the main theorem and which have some independent interest. Section 4 contains the proof of the main theorem.

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## 2. Preliminaries

Consider a bounded, connected domain $\Omega \subset \mathbb{R}^n$, with $\partial\Omega$ locally a Lipschitz graph with Lipschitz constant $L$. In later sections, $\Omega$ will be convex, but for the basic properties proved in this section convexity is not necessary. Let $\nu$ be the outer unit normal to $\partial\Omega$, which is well-defined almost everywhere on $\partial\Omega$.

Let

$$J[v] = \int_\Omega \left(|\nabla v|^2 + Q^2(x)\chi_{\{v>0\}}\right) dx,$$

where $\chi_D$ represents the characteristic function of the set $D \subset \Omega$, and $Q(x)$ is a measurable function with $0 \le Q(x) \le M$ for almost every $x \in \overline{\Omega}$. Let $S$ be a closed, proper, nonempty subset of $\partial\Omega$. Let $u_0$ be a smooth, nonnegative function on $\mathbb{R}^n$. Let $A = \sup_S u_0$. We minimize $J$ over the set

$$K = \{v \in H^1 : v = u_0 \text{ on } S\}.$$

We will use the following notation. Let $a > 0$. Then $\Omega_a$ will be the open set $\{x \in \Omega : d(x, S) > a\}$. Let $D \subset \mathbb{R}^n$ be a domain. For a function $f \in H^1(D)$, and a measurable set $T \subset \partial D$ of positive $n-1$-dimensional Hausdorff measure, $f|_T$ will denote the trace of the function $f$ along $T$, which is in $L^2(T)$. The support of an $H^1$ function $f$ will be denoted by $\mathrm{supp}(f)$ . Finally, $|D|$ denotes the $n$-dimensional Lebesgue measure of $D$ and

$$\fint_D f \mathrm{d}x := \frac{1}{|D|} \int_D f \mathrm{d}x$$

is the average of $f$ over $D$.

In this paper we are interested in harmonic functions satisfying Neumann boundary conditions in a weak sense.

**Definition 1.** *We say that a harmonic function $v$ on a Lipschitz domain $D$ satisfies Neumann boundary conditions weakly along an open set $\Gamma \subset \partial D$ if*

$$\int_\Omega \nabla v \cdot \nabla \phi \; dx = 0$$

*for every $\phi \in H^1(\Omega)$. The test functions may satisfy a Dirichlet boundary condition away from $\Gamma$.*

Note that this concept of Neumann boundary conditions is local, in that the behavior of $v$ away from a neighborhood around $\Gamma$ is irrelevant, and if it is proved to hold for a collection of open sets $\Gamma_i \subset \partial D$ such that $\bigcup_i \Gamma_i = \Gamma$ then it holds on $\Gamma$.

Let $u$ be a (local) minimizer of $J$ in $K$. We begin by listing some basic properties of $u$, as proved in [1], Lemmas 2.2-2.4.

**Lemma 1.** *A minimizer $u$ exists, and for any such minimizer (or local minimizer):*

(1) $-\Delta u \leq 0$ *on $\Omega$, in a distributional sense.*
(2) $0 \leq u \leq A$.
(3) $\forall\ \Omega' \Subset \Omega$, $\forall\ 0 < \alpha < 1$, $u \in C^{0,\alpha}(\Omega')$.
(4) $\forall\ \Omega' \Subset \Omega$, $\{u > 0\}$ *is open in* $\Omega'$.
(5) $\Delta u = 0$ *in* $\{u > 0\}$.

The minimizer $u$ is also Hölder continuous up to the boundary, although the exponent $\alpha$ is now not arbitrary, but instead is controlled by the Lipschitz constant of $\partial\Omega$. For this lemma, as with the main theorem, only points of $\Omega$ which are far from $S$ are considered.

**Lemma 2.** *Let $r_0 > 0$. Then $\exists\, \alpha > 0$ such that $u \in C^{0,\alpha}(\Omega_{r_0})$, with $\alpha$ depending on $n$, and $L$. $\|u\|_{C^\alpha}$ depends on $n, L, M$, $A$, and $r_0$.*

Proof There exists $s > 0$ depending only on the Lipschitz character of $\Omega$ such that one can cover $\partial\Omega$ with balls of radius $s$ and such that for each such ball $B$, $B \cap \partial\Omega$ is a Lipschitz graph with Lipschitz constant less than or equal to $L$. Let $r = \frac{1}{2}\min(1, r_0, s)$. Cover $\Omega$ with a finite number of balls of radius $r$ such that for each ball $B_r(x)$, $B_{2r}(x)$ is either an interior ball in $\Omega$ or $\partial\Omega \cap B_{2r}(x)$ is a Lipschitz graph as above. Let $x_0$ be the center of one of these balls. We will show that $u \in C^\alpha(B_r(x_0) \cap \Omega)$ for some $\alpha$ independent of $x_0$, from which we conclude the same for all of $\Omega_{r_0}$.

First, suppose $B_{2r}(x_0) \subset \Omega$. Then $u \in C^{0,\alpha}(B_r)$ for any $\alpha < 1$, as in Lemma 1. We may therefore suppose that $B_{2r}(x_0) \not\subset \Omega$. Let $x \in B_r(x_0)$ and let $r_x = \mathrm{dist}(x, \partial B_{2r}(x_0))$. Let $t < r_x$. Let $D_t = B_t(x) \cap \Omega$. Let $\Gamma_{D,t} = (\partial B_t(x)) \cap \Omega$, $\Gamma_{N,t} = B_t(x) \cap \partial\Omega$. If $\Gamma_{N,t} = \emptyset$, let $v_t$ be the harmonic function on $D_t$ such that $v_t = u$ on $\partial D_t$. In this case, as in [3], Theorem 2.1, we may conclude from the minimality of $u$ that, for each $t < r_x$,

$$\int_{D_t} |\nabla(u - v_t)|^2 \leq C(r_x) t^n.$$

Moreover, we may conclude that

(2) $$\int_{D_t} |\nabla(v_{2t} - v_t)|^2 \leq C(r_x)t^n.$$

as well.

If $\Gamma_{N,t} \neq \emptyset$, define $v_t$ to be the harmonic function on $D$ with $v_t = u$ on $\Gamma_{D,t}$ and $\frac{\partial v}{\partial \nu} = 0$ weakly along $\Gamma_{N,t}$. As above, for each $t < r_x$,

$$\int_{D_t} |\nabla(u - v_t)|^2 \leq C(r_x)t^n.$$

Now, because $\partial\Omega$ is a Lipschitz graph, there is a bilipschitz map

$$F : D_t \longrightarrow B_t^+(0)$$

with $\{x : x_n = 0\} = F(\Gamma_{N,t})$ and $(\partial B_{N,t}(0))^+ = F(\Gamma_{D,t})$. The Lipschitz constants of $F$ and $F^{-1}$ are controlled solely by $L$. Let

(3) $$\left(a^{ij}(y)\right) = |\det(\nabla F^{-1}(y))| \left((\nabla F)^{\mathrm{T}} \nabla F\right) (F^{-1}(y)),$$

whenever $y_n \geq 0$. When $y_n < 0$, let

(4) $$\left(a^{ij}(y_1, \ldots, y_n)\right) = \begin{cases} a^{ij}(y_1, \ldots, -y_n) & i \neq n \text{ and } j \neq n \\ -a^{ij}(y_1, \ldots, -y_n) & i = n \text{ or } j = n \text{ but } i \neq j \\ a^{ij}(y_1, \ldots, -y_n) & i = j = n \end{cases}.$$

Note that $a^{ij}$ is uniformly elliptic with bounded, measurable coefficients on all of $B_t(0)$, and that the bounds on $a^{ij}$ depend only on $L$.

Define $\tilde{v}_t(y) = v_t(F^{-1}y)$ on $B_t$ when $y_n \geq 0$. For $y \in B_t$ with $y_n < 0$, let $\tilde{v}\,(y_1, \ldots, y_n) = \tilde{v}\,(y_1, \ldots, -y_n)$.

Then, on $B_t(0)$, $\tilde{v}_t$ satisfies the equation

$$\sum_{i,j} \frac{\partial}{\partial x_j}(a^{ij} \frac{\partial \tilde{v}_t}{\partial x_i}) \mathrm{d}x = 0$$

in the weak sense, i.e. for every $\phi \in C_c^\infty(B_t(0))$,

$$\int\limits_{B_t(0)} \langle a^{ij} \nabla \tilde{v}_t, \nabla \phi \rangle = 0.$$

Because $v_t$ satisfies an elliptic equation of the appropriate form, by Theorem 5.3.6 of [10], there exists $C, \mu_0$, with $0 < \mu_0 < 1$, depending only on $n$ and the $a^{ij}$ (which in turn depend only on $L$) such that, for any $l < t < r$,

(5) $$\|\nabla v_t\|_{L^2(B_l \cap \Omega)} \leq C \|\nabla v_t\|_{L^2(B_t \cap \Omega)} \left(\frac{l}{t}\right)^{(\frac{n}{2} - 1 + \mu_0)}.$$

Now, we return to the consideration of $u$. Choose some $t < r_x$. Recall that we have

$$\int\limits_{B_t \cap \Omega} |\nabla(u - v_t)|^2 \mathrm{d}x \leq C t^n,$$

and note that this implies that

$$\int\limits_{B_{2^{(i-1)}t} \cap \Omega} |\nabla(v_{2^{i-1}t} - v_{2^i t})|^2 \mathrm{d}x \leq C(2^{i-1}t)^n.$$

Applying the result of our previous calculation to the function $v_{2^{i-1}t} - v_{2^i t}$ on $B_{2^{(i-1)}t} \cap \Omega$, we find that, if $\Gamma_{N,2^{i-1}t} \neq \emptyset$,

$$\int\limits_{B_t\cap\Omega} |\nabla(v_{2^{i-1}t} - v_{2^i t})|^2 \mathrm{d}x \leq Ct^n(2^{(2-2\mu_0)i}).$$

Recall that if $\Gamma_{N,2^{i-1}t} = \emptyset$, then we get the bound (2), which is even better.

We have, by the triangle inequality,

$$\begin{aligned} \|\nabla u\|_{L^2(B_t\cap\Omega)} &\leq \|\nabla(u-v_t)\|_{L^2(B_t\cap\Omega)} + \sum_{i=1}^{-\log(\frac{t}{r_x})} \|\nabla(v_{2^{i-1}t} - v_{2^i t})\|_{L^2(B_t\cap\Omega)} + \|\nabla v_r\|_{L^2(B_t\cap\Omega)} \\ &\leq Ct^{\frac{n}{2}}(1 + \sum_{i=1}^{-\log(\frac{t}{r_x})} (2^{(1-\mu_0)i})) + t^{(\frac{n}{2}-1+\mu_0)} \end{aligned}$$

After summation, we conclude that, for every $x \in B_r(x_0)$, for every $t < r_x$

$$\int\limits_{B_t(x)\cap\Omega} |\nabla u|^2 \mathrm{~d}x \leq C(r_x)t^{n-2+2\mu_0}.$$ [1]

Therefore, by Theorem 3.5.2 of [10], $u \in C^{\mu_0}(B_r(x_0)\cap\Omega)$, with $\mu_0$ and $\|u\|_{C^{\mu_0}}$ depending only on the given constants. □

Because $u$ is now known to be continuous on $\overline{\Omega}$, the following result is immediate. Recall that $S$ is closed in $\partial\Omega$, so $\Gamma = \partial\Omega \setminus S$ is open.

**Corollary 1.** $\{u > 0\} \cap \Gamma$ *is open in* $\partial\Omega$.

Finally, we consider the sense in which Neumann boundary conditions hold for $u$. Note that $\frac{\partial u}{\partial \nu}$ may not be defined pointwise along $\partial\Omega$, and in fact $\nu$ is not defined pointwise.

**Lemma 3.** $\frac{\partial u}{\partial \nu} = 0$ *weakly along* $\Gamma \cap \{u > 0\}$.

Proof Let $x_0 \in \Gamma$ with $u(x_0) > 0$. Choose $s > 0$ such that $u > 0$ on $B_s(x_0)\cap\Omega$, and $B_s(x_0)\cap S = \emptyset$. Let $\phi \in C_0^\infty(B_s(x_0))$. Since $u > 0$ on $B_s(x_0)$, $J$ is smooth for a general perturbation of $u$, and hence

$$0 = \int_{B_s(x_0)\cap\Omega} \nabla u \cdot \nabla\phi.$$

Since this holds for all such $x_0$ and $s$, we conclude that $\frac{\partial u}{\partial \nu} = 0$ weakly along $\Gamma$ in the sense defined by Definition 1. □

Finally note that, since $u$ is harmonic where it is positive, Lemma 4 in the next section will prohibit $u$ from being 0 at a point $x_0 \in \Gamma$ unless $B_r(x_0) \cap \Omega \cap \{u = 0\} \neq \emptyset$ for all $r > 0$. Additionally, if $u = 0$ in a neighborhood of $x_0$, then obviously $\frac{\partial u}{\partial \nu}$ is 0 there. So the only place in $\Gamma$ where Neumann boundary conditions for $u$ might possibly not make sense is at the free boundary interface itself.

## 3. Properties of Harmonic Functions

The proof of the main theorem will use several properties of harmonic functions on convex domains with Neumann boundary conditions, which are proven in this section. In Lemma 4, we prove a weak maximum principle for harmonic functions with mixed boundary conditions on a Lipschitz domain. We also present a Harnack inequality up to the Neumann boundary in Lemma 5. The section concludes with up-to-the-Neumann-boundary gradient bounds for harmonic functions in convex domains, in Lemmas 6 and 7.

[1] Note that, if $\mu_0 = 1$ then when we sum we will get $C(r_x)t^n|log(\frac{t}{r})|$, instead of the quantity computed above. This argument therefore cannot be used to obtain the estimate with $\mu_0 = 1$, i.e. Lipschitz continuity.

**Lemma 4.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain, and let $\Gamma_D \subset \partial\Omega$ be a measurable set of positive Hausdorff measure in $\partial\Omega$. Let $u \in H^1(\Omega)$ satisfy:*

(1) $\int_\Omega \nabla u \cdot \nabla \phi \geq 0 \qquad \forall \phi \in P = \{f \in H^1 | f \geq 0 \text{ and } f|_{\Gamma_D} = 0\}$

(2) $\exists\, u_0 \in L^2(\partial\Omega)$ *such that* $u|_{\Gamma_D} = u_0 \geq 0$ *on* $\Gamma_D$.

*Then $u \geq 0$ in $\Omega$.*

Proof Consider the function $u^-(x) = \max(0, -u(x))$. $u^-$ is in $H^1(\Omega)$ and $u^-|_{\Gamma_D} = 0$. Therefore, $u^- \in P$. Hence,

$$\int_\Omega \nabla u \cdot \nabla u^- \geq 0.$$

and we may conclude that

$$-\int_\Omega |\nabla u^-|^2 \geq 0$$

which implies that $\nabla u^- \equiv 0$, so, since $\Omega$ is connected, $u^-$ is constant. But $u^-|_{\Gamma_D} \equiv 0$, so $u^- \equiv 0$. Therefore, $u \geq 0$. $\square$

**Lemma 5.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, with Lipschitz constant $L$. Suppose $\Gamma_N$ is an open subset of $\partial\Omega$. Let $u$ be a positive harmonic function on $\Omega$ such that $\frac{\partial u}{\partial \nu} = 0$ along $\Gamma_N$, in a weak sense. Then, for any $x_0 \in \Omega \cup \Gamma_N$, for any $r > 0$ such that $B_r(x_0) \cap \partial\Omega \subset \Gamma_N$,*

$$\sup_{B_{\frac{r}{2}}(x_0)} u(x) \leq C \inf_{B_{\frac{r}{2}}(x_0)} u(x)$$

*where the constant $C$ depends only on $n$ and the Lipschitz character of $\partial\Omega$.*

Proof

Since $\Omega$ is bounded and the Harnack inequality holds in the interior of $\Omega$, it suffices to proof this inequality locally near $\Gamma_N$. Hence, we may choose $x_0 \in \Gamma_N$ and $s > 0$ such that:

(1) $B_{2s}(x) \cap (\partial\Omega \setminus \Gamma_N) = \emptyset$

(2) $\Gamma_N$ is simply connected in $B_{2s}$ and

(3) $\Gamma_N$ is a Lipschitz graph in the $x_n$-direction in $B_{2s}$, possibly after a rotation of coordinates.

Then, as in Lemma 2 there exists a bilipschitz map $F$ from $B_{2s}(x_0) \cap \Omega$ to $B^+_{2s}(0)$ such that $F$ extends continuously to the boundary and $F(\Gamma_N) = \{y | |y| \leq 2s \text{ and } y_n = 0\}$. The Lipschitz constants of $F$ and $F^{-1}$ depend only on the Lipschitz constant $L$ of $\Gamma_N$. As in Lemma 2, the function $\tilde{u} = u(F^{-1}y)$ on $B_{2s} \cap \{y_n \geq 0\}$ and defined by even reflection for negative $y_n$ satisfies a uniformly elliptic equation of divergence form with coefficients given by (3) and (4). Then, by ([9], Theorem 8.20), $\tilde{u}$ satisfies

$$\sup_U \tilde{u} \leq C \inf_U \tilde{u},$$

for any $U \Subset B_{2s}(0)$, where $C$ depends only on the eigenvalues of $a^{ij}$ and the distance from $\partial U$ to $\partial B_{2s}$. Clearly, the same inequality holds when we restrict to the upper half ball because $\tilde{u}$ was created by an even reflection. Choose $U = F(B_s(x))$. Then, by returning to $B_{2s}(x) \cap \Omega$ along $F^{-1}$, we conclude that $\sup_{B_s(x) \cap \Omega} u \leq C \inf_{B_s(x) \cap \Omega} u$, where $C$ depends only on $n$ and the Lipschitz constant $L$ of $\partial\Omega$. $\square$

We now provide a gradient control lemma for positive harmonic functions near a Neumann boundary. The function $\Phi$ was introduced in [6], and the idea for using convexity to control the Neumann boundary is similar to that used in [11]. The result is proven first on smooth domains, then a limiting procedure generalizes it to all convex domains.

**Lemma 6.** *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega$ is the graph of a smooth, convex function $f$. Suppose $0 \in \Omega$ and let $r = \text{dist}(0, \partial\Omega)$. Let $R > 2r$ and let $D = B_R(0) \cap \Omega$. Let $\Gamma = B_R \cap \partial\Omega$ and let $S = \overline{\partial B_R \cap \Omega}$. Let $u$ be a nonnegative harmonic function on $D$ bounded by a constant $A$, with $\frac{\partial u}{\partial \nu} = 0$ along $\Gamma$. Then $\exists C > 0$ depending only on $n$ such that $|\nabla u| \leq C\frac{A}{R}$ on $B_{\frac{R}{2}}$.*

Proof Clearly, we may assume that $u$ is not constant or else the lemma is trivially verified.

Let

$$\Phi(x) = \frac{(R^2 - |x|^2)^2\ |\nabla u|^2}{(9A^2 - (u - 2A)^2)^2}. \tag{6}$$

Then:

(1) $\Phi = 0$ on $S$.
(2) $\Phi > 0$ inside $D$.
(3) $\Phi$ is smooth in $D \cup \Gamma$ because $u$ and $\nabla u$ are smooth and the denominator of $\Phi$ cannot approach 0.
(4) $\max_{x\in\Gamma} \Phi(x) < \max_{x\in\overline{D}} \Phi(x)$. Note that $\Gamma$ is open.

Proof

$$\begin{aligned}\frac{\partial \Phi}{\partial \nu} &= 2\left(\frac{(R^2 - |x|^2)^2}{(9A^2 - (u - 2A)^2)^2}\right)(\nabla u \cdot \frac{\partial}{\partial \nu}\nabla u) \\ &\quad - 2\left(\frac{|\nabla u|^2(R^2 - |x|^2)}{(9A^2 - (u - 2A)^2)^2}\right)(\vec{x} \cdot \frac{\partial \vec{x}}{\partial \nu}) \\ &\quad - 2\left(\frac{|\nabla u|^2(R^2 - |x|^2)^2}{(9A^2 - (u - 2A)^2)^3}\right)(u - 2A)\frac{\partial u}{\partial \nu} \\ &= (a) + (b) + (c)\end{aligned}$$

Note that $(c) = 0$ on $\Gamma$ because $\frac{\partial u}{\partial \nu} = 0$. By the convexity of $\Omega$, $\vec{x} \cdot \frac{\partial \vec{x}}{\partial \nu} > 0$, so $(b)\ < 0$. Finally, consider $(a)$. Since $2\frac{(R^2-|x|^2)^2}{(9A^2-(u-2A)^2)^2} > 0$, we consider only $\nabla u \cdot \frac{\partial}{\partial \nu}\nabla u$, at a point $x$. After rotation, suppose that $\nu(x) = e_n$, so we can use $e_1, \ldots, e_{n-1}$ as local coordinates for $\Gamma$. Then

$$\nabla u \cdot \frac{\partial}{\partial \nu}\nabla u = \nabla u \cdot \nabla(\frac{\partial u}{\partial \nu}) - \sum_{i,j=1}^{n-1} \frac{\partial \nu_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = -\sum_{i,j=1}^{n-1} \frac{\partial \nu_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

because $\frac{\partial u}{\partial \nu} = 0$. But the matrix $\frac{\partial \nu_i}{\partial x_j}$ is just the second fundamental form of $\Gamma$ in local coordinates, and therefore by convexity it is positive definite. So $\nabla u \cdot \frac{\partial}{\partial \nu}\nabla u < 0$ along $\Gamma$. We conclude that $\frac{\partial \Phi}{\partial \nu} < 0$ along $\Gamma$, so the maximum of $\Phi$ cannot occur there. □

By (1) and (4), the maximum of $\Phi$ occurs at a point $x_0 \in D$. At $x_0$, we have:

$$0 = \nabla\Phi = \left(\frac{-2\nabla(|x|^2)}{(R^2 - |x|^2)} + \frac{\nabla(|\nabla u|^2)}{|\nabla u|^2} + \frac{2\nabla((u - 2A)^2)}{(9A^2 - (u - 2A)^2)}\right)\Phi \tag{7}$$

and

$$\begin{aligned}0 \geq \Delta\Phi = \Big(&\frac{-2\Delta(|x|^2)}{(R^2 - |x|^2)} + \frac{-2|\nabla(|x|^2)|^2}{(R^2 - |x|^2)^2} + \frac{\Delta(|\nabla u|^2)}{|\nabla u|^2} - \frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4} \\ &+ \frac{2\Delta((u - 2A)^2)}{(9A^2 - (u - 2A)^2)} + \frac{2|\nabla((u - 2A)^2)|^2}{(9A^2 - (u - 2A)^2)^2}\Big)\Phi\end{aligned} \tag{8}$$

Note that $\nabla(|x|^2) = 2\vec{x}$ and $\nabla((u - 2A)^2) = 2(u - 2A)\nabla u$. Plugging into (7), we have

$$0 = \frac{-4\vec{x}}{R^2 - |x|^2} + \frac{\nabla(|\nabla u|^2)}{|\nabla u|^2} + \frac{4(u - 2A)\nabla u}{(9A^2 - (u - 2A)^2)}$$

which implies that

$$\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4} \le \frac{16|x|^2}{(R^2-|x|^2)^2} + \frac{16(u-2A)^2|\nabla u|^2}{(9A^2-(u-2A)^2)^2} + \frac{32|x||u-2A||\nabla u|}{(R^2-|x|^2)(9A^2-(u-2A)^2)} \tag{9}$$

Because $\Delta u = 0$, we have $\Delta(|\nabla u|^2) = 2\sum_{i,j}(\frac{\partial^2 u}{\partial x_i \partial x_j})^2$, and

$$|\nabla(|\nabla u|^2)|^2 = 4\sum_{i,j,k} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_k}.$$

Therefore, comparing $2|\nabla u|^2\Delta(|\nabla u|^2)$ with $|\nabla(|\nabla u|^2)|^2$, we have:

$$\frac{\Delta(|\nabla u|^2)}{|\nabla u|^2} \ge \frac{1}{2}\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4}. \tag{10}$$

We also have $\Delta(|x|^2) = 2n$ and $\Delta((u-2A)^2) = 2|\nabla u|^2$. Plugging these and (10) into 8, we get:

$$\begin{aligned} 0 &\ge \frac{-4n}{R^2-|x|^2} - \frac{8|x|^2}{(R^2-|x|^2)^2} - \frac{1}{2}\frac{|\nabla(|\nabla u|^2)|^2}{|\nabla u|^4} + \frac{4|\nabla u|^2}{9A^2-(u-2A)^2} + \frac{8(u-2A)^2|\nabla u|^2}{(9A^2-(u-2A)^2)^2} \\ &\ge \frac{-4n}{R^2-|x|^2} - \frac{8|x|^2}{(R^2-|x|^2)^2} - \frac{8|x|^2}{(R^2-|x|^2)^2} - \frac{8(u-2A)^2|\nabla u|^2}{(9A^2-(u-2A)^2)^2} + \\ &\quad - \frac{16|x||u-2A||\nabla u|}{(R^2-|x|^2)(9A^2-(u-2A)^2)} + \frac{4|\nabla u|^2}{9A^2-(u-2A)^2} + \frac{8(u-2A)^2|\nabla u|^2}{(9A^2-(u-2A)^2)^2} \end{aligned}$$

Therefore,

$$\frac{4|\nabla u|^2}{9A^2-(u-2A)^2} \le \frac{16|x||u-2A||\nabla u|}{(R^2-|x|^2)(9A^2-(u-2A)^2)} + \frac{4n(R^2-|x|^2)+16|x|^2}{(R^2-|x|^2)^2}.$$

Multiplying by $(R^2-|x|^2)^2$, dividing by $9A^2-(u-2A)^2$, and recalling the definition of $\Phi$ given in (6), we have:

$$4\Phi(x_0) \le \frac{48RA\sqrt{(\Phi(x_0))}}{9A^2-(u-2A)^2} + \frac{(4n+16)R^2}{9A^2-(u-2A)^2}$$

Note that $5A^2 \le 9A^2-(u-2A)^2 \le 8A^2$. Let $z = \sqrt{\Phi(x_0)}$. Then the quadratic formula applied to $z$ implies that there exists a constant $C$ depending only on $n$ such that

$$\Phi(x_0) \le C\frac{R^2}{A^2}.$$

Since $x_0$ is the maximum of $\Phi$, we infer that

$$\Phi(x) \le C\frac{R^2}{A^2}$$

on all of $B_R(0)\cap\Omega$. Hence, on $B_{\frac{R}{2}}(0)\cap\Omega$, where $R^2-|x|^2 \sim R^2$, we find

$$|\nabla u|^2 \le c\frac{A^2}{R^2}.$$

for some $c$ depending only on $n$. □

**Lemma 7.** *Suppose that $\Omega$, $r$, $R$, $D$, $\Gamma$, $S$, and $u$ are as in Lemma 6, except that $\partial\Omega$ is no longer required to be smooth, only convex. Then $\exists C > 0$ depending only on $n$ such that $|\nabla u| \le C\frac{A}{R}$ on $B_{\frac{R}{2}}$.*

Proof Let $\Omega_i$ be a nested collection of smooth, convex domains contained in $\Omega$, such that $\lim_{i\to\infty}\Omega_i = \Omega$ and $0 \in \Omega_i$ for all $i$. Let $D_i = B_R(0)\cap\Omega_i$, and let $u_i$ be the function on $D_i$ satisfying

$$\begin{aligned} \Delta u_i(x) &= 0 && \forall x \in \Omega_i \\ u_i(x) &= u(x) && \forall x \in \partial B_R(0)\cap\Omega_i \\ \frac{\partial u_i}{\partial \nu}(x) &= 0 && \forall x \in B_R(0)\cap\partial\Omega_i \end{aligned}$$

Note that $0 \le \inf_{D_i} u_i$ and $\sup_{D_i} u_i \le \sup_D u = A$.

Then, there is a subsequence $u_{i_j}$ and a $u_0$ such that:

(1) $u_{i_j} \to u_0$ uniformly on $\overline{D\cap B_{\frac{3R}{4}}(0)}$.

By Theorem 5.3.7 of [10], $u_i \in C^\beta(D_i \cap B_{\frac{3R}{4}})$ for some $\beta > 0$ depending only on $L$ and moreover $\|u_i\|_{C^\beta}$ is bounded by a constant depending only on $n$, $L$, and $A$. Since the $u_i$ are uniformly bounded in $C^\beta$, a subsequence $u_{i_j}$ converges uniformly to a function $u_0$ on $D\cap B_{\frac{3R}{4}}(0)$.

(2) Due to interior regularity and (1), we may assume that the $u_{i_j}$ converge to $u_0$ in $C^\infty$ on compact subsets of $D\cap B_{\frac{3R}{4}}(0)$.

(3) We have that $\|\tilde{u_i}\|_{H^1(D)} \le C(n,B,R)$, which will allow us to assume that the sequence $\tilde{u}_{i_j}$ also converges weakly in $H^1(D)$.

Proof Recall that there exists a bounded extension operator from $H^1(D_i)$ to $H^1(D)$. Consider the function $\tilde{u}_i$ given by this extension of $u_i$ to $D$; $\|\tilde{u_i}\|_{H^1(D)} \le C\|u_i\|_{H^1(D_i)}$.

Consider $U_i = \Omega_i \cap B_{2R}$. Let $w_i$ be the minimizer of

$$\int_{U_i} |\nabla v|^2 dx$$

in $K_i = \{v \in H^1 : v = u \text{ on } \overline{U_i \setminus D_i}\}$. Then $w_i = u_i$ on $D_i$ because, $\forall \phi \in C_0^\infty(B_R)$,

$$\int_{D_i} (\nabla w_i \cdot \nabla\phi)dx,$$

so $w_i$ is harmonic and $\frac{\partial w_i}{\partial \nu} = 0$ on $\partial D_i \setminus \partial B_R$. In addition, $w_i = u$ on $\partial B_R \cap \Omega$, so $w_i = u_i$ on $D_i$. We conclude that $u_i$ satisfies

$$\int_{U_i} |\nabla u_i|^2 dx \le \int_{U_i} |\nabla u|^2 dx \le \|u\|_{H^1(B_{2R}\cap\Omega)}.$$

Since, in addition,

$$\int_{U_i} u_i^2 dx \le (2R)^n B^2,$$

we conclude that $\|u_i\|^2_{H^1(D_i)} \le C(n,B,R)$. □

Therefore, $\|\tilde{u_i}\|_{H^1(D)} \le C(n,B,R)$, so we may assume that the sequence $\tilde{u}_{i_j}$ also converges weakly in $H^1(D)$. Moreover, this weak limit function must also be the weak limit of the $\tilde{u}_{i_j}$ on any fixed $D_{i_0}$, but for $i_j > i_0$, $\tilde{u}_{i_j} = u_{i_j}$ on $D_{i_0}$. Therefore, since $u_{i_j} \to u_0$ uniformly on $D_{i_0}$, the weak limit of $\tilde{u}_{i_j}$ must also be $u_0$.

By interior $C^\infty$ convergence, we know $u_0$ is harmonic in $D$, and by construction $u_0 = u$ on $\partial B_R(0)\cap\Omega$. Moreover, for any $\phi \in C_0^\infty(B_R)$, we have:

$$\begin{aligned} \int_D \nabla u_0\cdot\nabla\phi &= \int_{D_{i_j}} (\nabla u_0 - \nabla u_{i_j})\cdot\nabla\phi \;+\; \int_{D_{i_j}} \nabla u_{i_j}\cdot\nabla\phi \;+\; \int_{D\setminus D_{i_j}} \nabla u_0\cdot\nabla\phi \\ &= (1)+(2)+(3) \end{aligned}$$

As $i_j \to \infty$, $(1)\to 0$ by weak-$H^1$ convergence of the $u_{i_j}$ to $u_0$. By construction of the $u_i$, $(2) = 0$. Finally, $(3) \le \|u_0\|_{H^1}\|\phi\|_{H^1}|D\setminus D_{i_j}| \to 0$ by construction of the $D_i$, since $\|u_0\|_{H^1} \le \|u\|_{H^1}$.

We conclude that $\frac{\partial u_0}{\partial \nu} = 0$ weakly along $B_R \cap \partial\Omega$. Hence, by the uniqueness of harmonic functions with this type of domain and boundary condition (from the maximum principle, Lemma 4), $u_0 = u$ on $D$. So, $u$ is the uniform limit of the $u_{i_j}$ on $D \cap B_{\frac{3R}{4}}(0)$. Note that, by Lemma 6, the $u_{i_j}$ satisfy the gradient bound

$$|\nabla u_{i_j}|^2 \leq c\frac{A^2}{R^2}$$

for each $i_j$ on $D \cap B_{\frac{R}{2}}(0)$. Therefore, by uniform convergence, $u$ also satisfies the bound

$$|\nabla u|^2 \leq c\frac{A^2}{R^2}.$$

almost everywhere on $D \cap B_{\frac{R}{2}}(0)$. □

These two lemmata also hold if $u$ is not nonnegative, with $A$ replaced by the total variation of $u$ on $D$.

Note that this lemma has some interest in its own right, as a boundary regularity result for harmonic functions. One example of an application of this is to the size of the first nontrivial Neumann eigenvalue of the spherical laplacian on (geodesically) convex subsets of the sphere. Let $f(\theta)$ be such an eigenfunction on $V \subset S^{n-1}$, with eigenvalue $\lambda$. Let $\alpha$ be given by $\alpha(\alpha + n - 2) = \lambda$. Then the function $\tilde{f} = |x|^\alpha f(\frac{x}{|x|})$ is harmonic in the set $\tilde{V} = \{x \in \mathbb{R}^n : |x| \leq 1, \frac{x}{|x|} \in V\}$. The geodesic convexity of $V$ implies that $\tilde{V}$ is convex in $\mathbb{R}^n$, and the Neumann boundary conditions on $V$ in $S^{n-1}$ correspond to Neumann boundary conditions for $\tilde{f}$ along $\partial\tilde{V} \cap \{x : |x| < 1\}$. Therefore, the above lemma applies to $\tilde{f}$ on $\tilde{V}$, so $\tilde{f}$ must be Lipschitz up to the boundary on $\frac{1}{2}\tilde{V}$, including the origin. But Lipschitz continuity of $\tilde{f}$ at 0 is equivalent to $\alpha \geq 1$, which is equivalent to $\lambda \geq n - 1$.

This lower bound for eigenvalues is known on manifolds of positive Ricci curvature with convex boundary ([12], [5]). However, as the above lemma is proved using a maximum principle argument, it represents a new, entirely non-variational proof of this result for the special case of the sphere.

## 4. Lipschitz Regularity for the Free Boundary Problem

This section contains the proof of the main result of this paper, namely that solutions of the free boundary problem are Lipschitz continuous up to convex Neumann boundaries. The problem is as defined in Section 2. The main tool is a lemma that gives an average growth rate of $u$ away from the free boundary which is compatible with Lipschitz regularity. This result follows from a generalization of the techniques used in [1], so that they apply close to a convex boundary with Neumann boundary conditions. In particular, we adapt the argument of [1] Lemma 3.2. The next step is to prove Lipschitz regularity on the Neumann boundary itself, which is done by application of the gradient control result of the previous section. Using these tools, we give a complete proof of Lipschitz continuity via the maximum principle.

**Lemma 8.** *There is a $C$ depending only on $n, L$, and $M$ such that $\forall x \in \partial\Omega$, $\forall B_r(x) \subset \mathbb{R}^n$ such that $B_{2r}(x) \cap S = \emptyset$ and $B_r(x) \cap \partial\Omega$ is a Lipschitz graph,*

$$\frac{1}{r}(u(x)) > C \Rightarrow u > 0 \text{ in } B_r(x) \cap \Omega.$$

Proof Without loss of generality, we may assume that $x = 0$. Define $D = B_r(0) \cap \Omega$. Let $\Gamma_D = \partial B_r(0) \cap \Omega$ and $\Gamma_N = B_r(0) \cap \partial\Omega$. Then, let $v \in H^1(B_r \cap \Omega)$ be the harmonic function satisfying $v = u$ on $\Gamma_D$ and with weak Neumann boundary conditions along $\Gamma_N$. Note that $v$ minimizes the functional $\int_{B_r \cap \Omega} |\nabla f|^2$ on the set

$$K = \{ f \in H^1(D) | f = u \text{ on } \Gamma_D \}.$$

We can conclude by Lemma 4 that $v \geq 0$ on $B_r \cap \Omega$, and therefore, by the usual strong maximum principle, $v > 0$ on the interior.

Moreover, $v$ is a valid competitor for $u$ as minimizer of $J$, so:

$$\int_D (|\nabla v|^2 + Q^2) \geq \int_D (|\nabla u|^2 + Q^2 \chi_{\{u>0\}}),$$

which implies that

$$\int_D |\nabla(v-u)|^2 \leq \int_D Q^2 \chi_{\{u=0\}}. \tag{11}$$

Now, we need to obtain an estimate in the opposite direction. Namely, we claim that

$$(\frac{1}{r}u(x))^2 \int_D \chi_{\{u=0\}} \leq C \int_D |\nabla(v-u)|^2. \tag{12}$$

Comparing this estimate with (11) will imply the claim of the lemma.

Note that if we dilate by the formula $u_r(y) = \frac{1}{r}u(ry)$, and similarly for $v$,then both sides of (12) are unaffected, so we may assume that $r = 1$, i.e. we may assume that $D = B_1(0) \cap \Omega$. In addition, we assume (possibly after a rotation) that $\partial\Omega \cap B_1(0)$ is a Lipschitz graph in the $x_n$-direction, with Lipschitz constant $L$. Then, there exists an $\epsilon(L)$ such that $B_{2\epsilon}(0, 0, \ldots, 0, \frac{1}{2}) \subset D$. Note that $\epsilon \leq \frac{1}{4}$. Because $D$ is convex, for each $z \in B_\epsilon((0, 0, \ldots, 0, \frac{1}{2}))$, $D$ is star-shaped with respect to $z$.

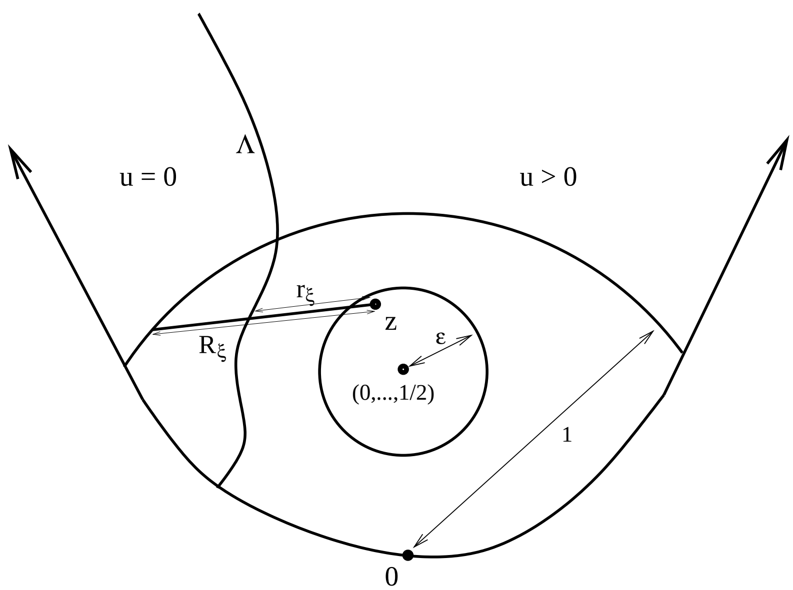


Figure 1. The shape of the domain near the intersection of free and fixed boundary

Note that since $0 \in \partial\Omega$ and $\Omega$ is convex, $\partial\Omega \cap B_1(0)$ is simply connected, and $\Omega \cap B_1(0)$ is contained in $\{x \in B_1(0) : x_n > 0\}$. Let $F$ be a bilipschitz map from $D = \Omega \cap B_1(0)$ to $D' = B_1(0) \setminus \overline{D}$, such that $F$ extends continuously to a map from $\overline{D}$ to $\overline{D'}$ with $F\Big|_{\partial\Omega} = \mathrm{Id}$, and $F(\Omega \cap \partial B_1(0)) = (\partial B_1(0)) \setminus \Omega$ . The Lipschitz constants of $F$ and $F^{-1}$ depend only on $L$. Define the function $\tilde{u}$ on $B_1(0)$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \forall x \in \overline{\Omega} \cap B_1(0) \\ u(F^{-1}x) & \forall x \in B_1(0) \setminus \overline{\Omega} \end{cases}$$

and define $\tilde{v}$ similarly.

For every $\xi \in S^{n-1}$, we define

$$R_\xi = \sup\{r \,|\, r\xi + z \in B_1(0)\}$$
$$r_\xi = \inf\{r \,|\, \frac{\epsilon}{2} \le r \le R_\xi \text{ and } \tilde{u}(r\xi + z) = 0\}$$

If $\{r \,|\, \frac{\epsilon}{2} \le r \le R_\xi \text{ and } \tilde{u}(r\xi+z) = 0\} = \emptyset$, let $r_\xi = R_\xi$. Define $\tau_\xi(t) = z + t\xi$ for $r_\xi \le t \le R_\xi$, and denote the entire curve by $\tau_\xi$. Note that $\tilde{v}(\tau(R_\xi)) = \tilde{u}(\tau(R_\xi))$ because this point is on the boundary of $B_1(0)$. (If $y \in \partial(B_1(0) \setminus D)$, $F^{-1}(y) \in \partial(B_1(0) \cap D)$, so $\tilde{v}(y) = v(F^{-1}(y)) = u(F^{-1}(y)) = \tilde{u}(y)$ and we have $\tilde{v}(\tau(R_\xi)) = \tilde{u}(\tau(R_\xi))$ as before.) Also note that the path $\tau$ has unit speed at all times, and recall that $\tilde{u}(\tau(r_\xi)) = 0$ (unless $r_\xi = R_\xi$, in which case $|\tau_\xi| = 0$). Then

$$\begin{aligned}
\tilde{v}(r_\xi\xi + z) &= \tilde{v}(r_\xi\xi + z) - \tilde{u}(r_\xi\xi + z) \\
&= \tilde{v}(\tau(r_\xi)) - \tilde{u}(\tau(r_\xi)) \\
&= \tilde{v}(\tau(R_\xi)) - \tilde{u}(\tau(R_\xi)) - \int_{\tau_\xi} \frac{\partial}{\partial t}\left((\tilde{v} - \tilde{u})(\tau_\xi(t))\right) \mathrm{d}t \\
&= 0 + \int_{\tau_\xi} \frac{\partial}{\partial t}\left((\tilde{u} - \tilde{v})(\tau_\xi(t))\right) \mathrm{d}t \\
&\le \int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})| \mathrm{d}t \\
&\le \sqrt{|\tau_\xi|}\left(\int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \qquad (13)
\end{aligned}$$

Recall that $|\tau_\xi| = R_\xi - r_\xi$. Define $s_\xi$ to be the unique (due to convexity) $s < R_\xi$ such that $\tau(s) \in \partial\Omega$ if such an $s$ exists. Otherwise, let $s_\xi = R_\xi$.

Now we will estimate $\tilde{v}(r_\xi\xi + z)$ from below. We claim that there exists $c > 0$ so that $v(x) \ge c(1 - |x|)$ for all $x \in D$. We know that $v$ is harmonic on $B_{\frac{1}{4}}(0) \cap \Omega$ and moreover that this domain is far from any Dirichlet boundary pieces of $D$. Hence, by the modified Harnack inequality (Lemma 5), there exists $c > 0$ so that $v(x) \ge cv(0)$ for every $x$ in $B_{\frac{1}{8}}(0) \cap D$. Now, let $V = D \setminus B_{\frac{1}{8}}(0)$, and on $V$ define

$$H(x) = \frac{cv(0)}{1 - 8^{2-n}}(|8x|^{2-n} - 8^{2-n})$$

for $n > 2$ and

$$H(x) = \frac{-cv(0)}{\log 8} \log|x|$$

for $n = 2$. Then, on $V$, the function $v - H$ has the following properties:

$$\int_V \nabla(v - H) \cdot \nabla\phi \ge 0 \qquad \forall\phi \in \{f \in H^1(V) \,|\, f \ge 0 \text{ and } f = 0 \text{ on } (\partial B_{\frac{1}{8}}(0) \cup \partial B_1(0)) \cap \Omega)\}$$
$$v - H \ge 0 \qquad \text{on } \partial B_{\frac{1}{8}}(0) \cap \Omega \text{ and } \partial B_1(0) \cap \Omega$$

The first property holds because $v - H$ is weakly harmonic on $V$, and in addition $\frac{\partial H}{\partial \nu} < 0$ along $\partial V \cap (B_1(0) \setminus B_{\frac{1}{8}}(0))$ by the convexity of $D$, so $\frac{\partial(v-H)}{\partial\nu} \ge 0$ weakly there. Hence, by Lemma 4, $v - H \ge 0$ on $V$. Note that, for $x \in V$, there exists $c > 0$ so that $H(x) \ge cv(0)(1 - |x|)$, because $H$ reaches 0 with nontrivial derivative at $|x| = 1$ and $H$ is strictly decreasing in the radial variable. Hence, for $|x| > 1/8$, $v(x) \ge cv(0)(1 - |x|)$. Note also that, for $|x| \le 1/8$, the same holds simply because, by Lemma 5, $v(x) \ge cv(0) \ge cv(0)(1 - |x|)$. Therefore, $v(x) \ge cv(0)(1 - |x|)$ for all $x \in D$ as claimed.

If $r_\xi \le s_\xi$, we immediately conclude that

$$\tilde{v}(r_\xi\xi + z) \ge cv(0)(1 - |r_\xi\xi + z|).$$

On the other hand, if $r_\xi > s_\xi$, then $\tilde{v}(r_\xi \xi + z) = v(F^{-1}(r_\xi \xi + z)) \geq cv(0)(1 - |F^{-1}(r_\xi \xi + z)|)$. Because $F$ is a bilipschitz map, this implies that $\tilde{v}(r_\xi \xi + z) \geq cv(0)(1 - |r_\xi \xi + z|)$ for a perhaps smaller $c$. In either case, we have that $\tilde{v}(r_\xi \xi + z) \geq cv(0)(1 - |r_\xi \xi + z|)$.

Additionally, $1 - |r_\xi \xi + z| \geq c(R_\xi - r_\xi)$. To check this we may assume that $(1 - |r_\xi \xi + z|) < \frac{1}{4}$, because otherwise, since $R_\xi - r_\xi \leq 2$ we are trivially done, with $c = 1/8$.

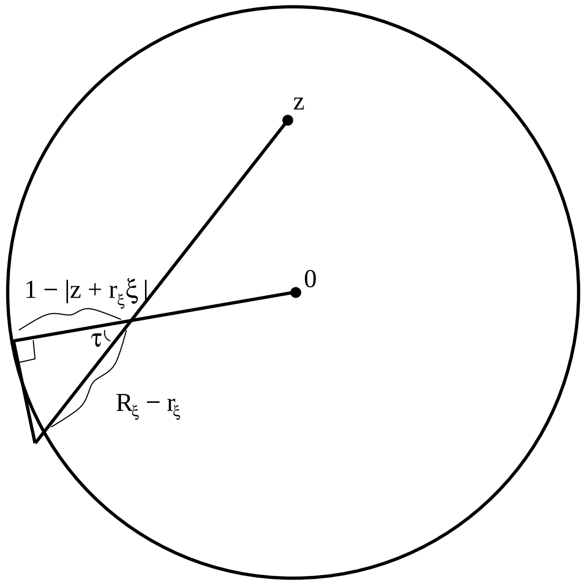


FIGURE 2. Comparision of $R_\xi - r_\xi$ and $1 - |r_\xi \xi + z|$.

Because $|z| \leq \frac{3}{4}$, there exists $\tau_0 < 90°$ such that if $|r_\xi \xi + z| > \frac{3}{4}$, the ray from 0 to the point $r_\xi \xi + z$ and the ray from $z$ to that point must meet at an angle $\tau < \tau_0$. But then, by definition of cosine on the triangle shown in the figure, $R_\xi - r_\xi \leq \frac{1}{\cos(\tau)}(1 - |r_\xi \xi + z|) \leq \frac{1}{\cos(\tau_0)}(1 - |r_\xi \xi + z|) \leq C(1 - |r_\xi \xi + z|)$. Therefore, we conclude that

$$cv(0)(R_\xi - r_\xi) \leq \tilde{v}(r_\xi \xi + z).$$

Combining this equation with (13), we obtain:

$$cv(0)(R_\xi - r_\xi) \leq C'(R_\xi - r_\xi)^{\frac{1}{2}}(\int_{\tau_\xi} |\nabla(v - u)|^2 \mathrm{d}t)^{\frac{1}{2}},$$

which implies that

$$C\, v(0)^2 (R_\xi - r_\xi) \leq \int_{\tau_\xi} |\nabla(\tilde{v} - \tilde{u})|^2 \mathrm{d}t \tag{14}$$

Integrating in $\xi$, we obtain, for the left-hand side of (14):

$$\begin{aligned} \int\limits_{S^{n-1}} (R_\xi - r_\xi)\mathrm{d}\xi &= \int\limits_{S^{n-1}} \int_{r_\xi}^{R_\xi} \mathrm{d}r\mathrm{d}\xi \geq \frac{1}{2^{n-1}} \int\limits_{S^{n-1}} \int_{r_\xi}^{R_\xi} r^{n-1}\mathrm{d}r\mathrm{d}\xi \\ &\geq \frac{1}{2^{n-1}} \int\limits_{B_1(0)\setminus B_{\frac{\varepsilon}{2}}(z)} \chi_{\{\tilde{u}=0\}}\mathrm{d}x \geq \frac{1}{2^{n-1}} \int\limits_{D\setminus B_{\frac{\varepsilon}{2}}(z)} \chi_{\{u=0\}}\mathrm{d}x. \end{aligned} \tag{15}$$

For the right-hand side of (14) we have to consider two cases. If $s_\xi \geq r_\xi$ we have:

$$\int_{r_\xi}^{s_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}r = \int_{r_\xi}^{s_\xi} |\nabla(v-u)(r\xi+z)|^2 \mathrm{d}r$$

and

$$\begin{aligned}\int_{s_\xi}^{R_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}r &\leq \int_{s_\xi}^{R_\xi} |\nabla F^{-1}|^2 |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}r \\ &\leq C(L) \int_{s_\xi}^{R_\xi} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}r.\end{aligned}$$

If $s_\xi < r_\xi$, we have

$$\begin{aligned}\int_{r_\xi}^{R_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}r \ &\leq \int_{r_\xi}^{R_\xi} |\nabla F^{-1}|^2 |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}r \\ &\leq C(L) \int_{r_\xi}^{R_\xi} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}x.\end{aligned}$$

In either case we have

$$\int_{r_\xi}^{R_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 dr \leq \int_{r_\xi}^{s_\xi} |\nabla(v-u)(r\xi+z))|^2 dr + C(L) \int_{s_\xi}^{R_\xi} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}r.$$

where we take the first term to be zero if $r_\xi \geq s_\xi$. Recall that, by definition, $r_\xi \geq \epsilon(L)/2$, and $s_\xi \geq 2\epsilon(L)$ by the domain geometry discussed above. Integrating in $\xi$, we get

$$\begin{aligned}\int\limits_{S^{n-1}} \int_{r_\xi}^{R_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 \mathrm{d}r\mathrm{d}\xi &\leq \frac{2^{n-1}}{\epsilon(L)^{n-1}} \int\limits_{S^{n-1}} \int_{r_\xi}^{s_\xi} |\nabla(v-u)(r\xi+z)|^2 r^{n-1} \mathrm{d}r\mathrm{d}\xi \\ &\quad + \frac{C(L)}{(2\epsilon(L))^{n-1}} \int\limits_{S^{n-1}} \int_{s_\xi}^{R_\xi} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 r^{n-1} \mathrm{d}r\mathrm{d}\xi\end{aligned}$$

Clearly,

$$\frac{2^{n-1}}{\epsilon(L)^{n-1}} \int\limits_{S^{n-1}} \int_{r_\xi}^{s_\xi} |\nabla(v-u)(r\xi+z)|^2 r^{n-1} \mathrm{d}r\mathrm{d}\xi \leq C(L) \int\limits_D |\nabla(v-u)|^2 \mathrm{d}x.$$

Moreover,

$$\begin{aligned}\frac{C(L)}{(2\epsilon(L))^{n-1}} \int\limits_{S^{n-1}} \int_{s_\xi}^{R_\xi} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 r^{n-1} \mathrm{d}r\mathrm{d}\xi &\leq C(L) \int\limits_{D'} |\nabla(v-u)(F^{-1}(r\xi+z))|^2 \mathrm{d}x \\ &\leq C(L) \int\limits_D |\nabla(v-u)(y)|^2 |Det(\nabla F(y))| \mathrm{d}y \\ &\leq C(L) \int\limits_D |\nabla(v-u)|^2 \mathrm{d}y,\end{aligned}$$

making the substitution $y = F^{-1}(x)$. Therefore, we conclude that

$$\int\limits_{S^{n-1}} \int\limits_{\tau_\xi} |\nabla(\tilde{v}-\tilde{u})|^2 dr d\xi \leq C(L) \int\limits_D |\nabla(v-u)|^2 \mathrm{d}x \tag{16}$$

which controls the right hand side of (14).

We combine (14), (15), and (16) to find that:

$$v(0)^2 \int\limits_{D\setminus B_{\frac{\epsilon}{2}}(z)} \chi_{\{u=0\}} \leq C \int\limits_D |\nabla(v-u)|^2.$$

Finally. we integrate over $z \in B_{\epsilon(L)}((0,\ldots,0,\frac{1}{2}))$ to conclude that

$$v(0)^2 \int\limits_D \chi_{\{u=0\}} \leq C(L) \int\limits_D |\nabla(v-u)|^2$$

which, when we combine with equation (11), yields:

$$v(0)^2 \int\limits_D \chi_{\{u=0\}} \leq C(n,L,M) \int\limits_D \chi_{\{u=0\}}.$$

and we conclude that if $v(0) > \sqrt{C(n,L,M)}$ then $\{u=0\}$ has measure zero in $D$. But then (11) implies that $u$ is identical with $v$ in $D$, i.e. $u$ is a positive harmonic function in $D$, so $u$ is strictly positive in $D$. Recall that, because $u$ is globally subharmonic, $u(0) \leq v(0)$, so we may conclude that, if $\{u=0\} \cap B_r \neq \emptyset$, then $u(0) \leq C$. □

The next step is to check the Lipschitz gradient bound on the fixed boundary, near the free boundary. This lemma depends upon the gradient estimate in Lemma 7 as well as the average growth control from Lemma 8.

**Lemma 9.** *Let $\Omega \subset R^n$ be a bounded, convex domain. Suppose that $x \in \partial\Omega$ with $d(x,\Lambda) < d(x,S)$, where $\Lambda = \partial\{u>0\}$ is the free boundary. Then there exists $C>0$ depending only on $n$, $L$, and $M$ so that*

$$|\nabla u(x)| \leq C.$$

Proof Define $r_x = \inf\{r>0 : B_r(x) \cap \{u=0\} \neq \emptyset\}$. Note that $u$ is a positive harmonic function on $B_{r_x}(x) \cap \Omega$. Therefore, by Lemma 5, there is a $C>0$ such that $\sup_{B_{\frac{r_x}{2}}}(x) \leq Cu(x)$.

Now, for any $\delta > 0$, $B_{r_x+\delta}(x) \cap \{u=0\}$ has positive measure. So by Lemma 8

$$u(x) \leq C(n,L,M)\frac{1}{r_x+\delta}.$$

Therefore,

$$u(x) \leq \frac{C(n,L,M)}{r_x}.$$

So, $\forall y \in B_{\frac{r_x}{2}}$,

$$u(y) \leq \frac{C(n,L,M)}{r_x}.$$

Now, $u$ is again a positive harmonic function on $B_{\frac{r_x}{2}} \cap \Omega$. Moreover, $0 \leq u \leq C(n,\ L,\ M)\ r_x$ on $B_{\frac{r_x}{2}} \cap \Omega$. We may therefore apply Lemma 7, with $A = C(n,L,M)r_x$ to conclude that

$$|\nabla u(x)| \leq C\frac{C(n,L,M)r_x}{r_x} \leq C(n,L,M)$$

as claimed. □

Finally, we arrive at the main theorem.

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $S$ be a closed subset of $\partial\Omega$ and let $\Gamma = \partial\Omega \setminus S$. Suppose $\partial\Omega$ is convex in a neighborhood of $\overline{\Gamma}$. Let $r_0 > 0$. Then there is a constant $C(n,L,r_0,M,A)$ such that for almost every $x \in \Omega_{r_0}$, $|\nabla u(x)| \leq C$.*

Proof

Recall that $U = \{ x \in \Omega | \, u(x) > 0 \}$ and let $\Lambda = \Omega \cap \partial U$ be the free boundary. Let $x \in \Omega_{r_0}$. There are five cases, as illustrated in Figure 4 (These cases may partially overlap.):

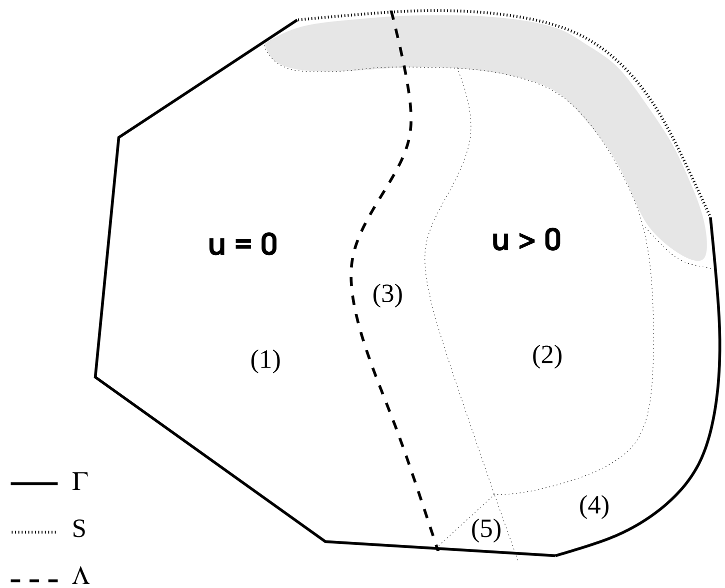


Figure 3. The subsets of the domain $\Omega$ which form the cases in Theorem 2

(1) $x \in (\Omega \setminus U)$. $|\nabla u| = 0$ for almost every such $x$.[2]
(2) $x \in U$ and $d(x, \partial U) \geq 1$. Then, by the interior regularity of harmonic functions on $B_1(x)$,

$$|\nabla u(x)| \leq \sup_{\Omega} u \leq \sup_{S} u = A.$$

(3) $x \in U$ and $d(x, \partial\Omega) > d(x, \Lambda)$. Then, let $r = d(x, \Lambda)$. As in ([1], Corollary 3.3), we can conclude that

$$|\nabla u(x)| \leq \frac{C}{r} \fint_{\partial B_r(x))} u \leq C(M).$$

(4) $x \in U$ and $d(x, \Lambda) \geq 1$. Let $r = \min(r_0, 1)$. Then $u$ is a positive, harmonic function on $B_r(x) \cap \Omega$, bounded by $A$. Therefore, by Lemma 7, $|\nabla u(x)| \leq C(n, L) \frac{A}{r_0}$.
(5) Finally, consider $x \in U \cap \Omega_{r_0}$ such that $d(x, \Gamma) \leq d(x, \Lambda) \leq 1$.

Recall from Case 3, that, $\forall x \in U$ with $\mathrm{dist}(x, \Lambda) < \mathrm{dist}(x, \partial\Omega)$, $|\nabla u| \leq C(n, M)$. Therefore, by slightly shrinking $U$ and applying Case 3, we can find a set $U'$ such that $x \in U'$, $\partial U' \cap \Omega$ is smooth, and $|\nabla u| \leq C(n, M)$ on $(\partial U') \cap \Omega$.

Since $u$ is in $H^1$, $\nabla u$ is in $L^2$. Therefore, by Fubini's Theorem, there exists a radius $r$ such that $\frac{3}{4} r_0 \leq r \leq r_0$ and $\nabla u \in L^2(\partial B_r(x) \cap U)$ with $r^{n-1} \|\nabla u\|_{L^2(\partial B_r(x))} \leq \|u\|_{H^1}$.

Consider, $D = B_r(x) \cap U'$. Then $u$ is a positive harmonic function on $D$. Moreover, $\partial D$ has three parts: $\Gamma_1 = B_r \cap \Gamma \cap \partial U'$, $\Gamma_2 = B_r \cap \Omega \cap \partial U'$, and $\Gamma_3 = U' \cap \partial B_r$. These parts may not each be connected, but they are disjoint and the union of their closures is the entire boundary of $\partial D$.

Note that $\Gamma_1$ is a convex Lipschitz hypersurface and, by Lemma 9, $|\nabla u| \leq C$ on $\Gamma_1$. Note also that $\Gamma_2$ is a Lipschitz curve on which $|\nabla u| \leq C$ by construction. Finally, $\Gamma_3$ is a smooth curve.

[2] $\Lambda$ itself has measure zero, so we do not need to consider $|\nabla u|$ there.

We define the function $v$ on $B_r(x)$ by:

$$\Delta v = 0 \qquad \text{on } B_r(x)$$
$$v = C^2 + (|\nabla u|)^2 \chi_{\Gamma_3} \qquad \text{on } \partial B_r(x).$$

Then, on $\Gamma_1$ and $\Gamma_2$, $v \geq C^2 \geq |\nabla u|^2$, and on $\Gamma_3$, $v = C^2 + |\nabla u|^2 > |\nabla u|^2$. Hence, since $|\nabla u|^2$ is subharmonic, by the maximum principle $v \geq |\nabla u|^2$ on $D \subseteq B_r$. So $v(x) > |\nabla u(x)|^2$.

But, using the Poisson kernel, we find that

$$\begin{aligned}
v(x) &= \int_{\partial B_r(x)} P * v(y)\, \mathrm{d}\sigma(y) \\
&= \fint_{\partial B_r(x)} v(y) \mathrm{d}\sigma(y) \\
&= C^2 + \frac{1}{\omega_n r^{n-1}} \left( \int_{\partial B_r(x)} v(y)\ \chi_{\Gamma_3}(y) \mathrm{d}\sigma(y) \right) \\
&\leq C^2 + \frac{1}{\omega_n r^{n-1}}\ \|v\|_{L^2(\Gamma_3)} |\Gamma_3|^{\frac{1}{2}} \\
&\leq C^2 + \frac{1}{\omega_n r^{n-1}} r^{n-1} \|\nabla u\|_{L^2(\Omega)} |\Gamma_3|^{\frac{1}{2}} \\
&\leq C^2(n, L, M, r_0, A)
\end{aligned}$$

And we conclude that $|\nabla u(x)| \leq C(n, L, M, r_0, A)$.

So, we can finally conclude that for almost every $x \in \Omega_{r_0}$, $|\nabla u(x)| \leq C(r_0, A, M, L)$, and, hence $u \in C^{0,1}(\Omega_{r_0})$. $\square$

## References


[1] H. Alt, and L. Caffarelli. " Existence and Regularity for a Minimum Problem with Free Boundary, " *J. Reine Angew. Math.* **325** (1981), 105–144.

[2] H. Alt, L. Caffarelli, and A. Friedman. " Jets With Two Fluids II, " *Indiana U. Math. J.* **33** (1984), 367-391.

[3] H. Alt, L. Caffarelli, and A. Friedman. " Variational Problems with Two Phases and Their Free Boundaries, " *Trans. Amer. Math. Soc.* **282** (1984), 431–461.

[4] H. Berestycki, L. Caffarelli, and L. Nirenberg. " Uniform Estimates for Regularization of Free Boundary Problems," *Analysis and Partial Differential Equations, Lecture Notes in Pure and Appl. Math.* **122** (1990), 567-619.

[5] X. Changyu. " The First Nonzero Eigenvalue for Manifolds with Ricci Curvature Having Positive Lower Bound, " *Chinese Mathematics into the Twenty-First Century (Tianjin, 1988)*, Peking University Press, Beijing, 1991, 243-249.

[6] S.-Y. Cheng. "Liouville Theorem for Harmonic Maps," *Proc. Sympos. Pure Math.* **36** (1980), 147-151.

[7] A. Friedman. *Variational Principles and Free Boundary Problems,* John Wiley & Sons, Inc., New York, 1982.

[8] A. Gurevich. " Boundary Regularity for Free Boundary Problems, " *Comm. Pure Appl. Math.* **52** (1999), 363–403.

[9] D. Gilbarg, and N Trudinger. *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, New York, 2001.

[10] C. Morrey. *Multiple Integrals in the Calculus of Variations,* Springer-Verlag, New York, 1966.

[11] P. Li, and S.-T. Yau. "On the Parabolic Kernel of the Schrödinger Operator", *Acta Math.* **156** (1986), 153-201.

[12] E. Pak, H. Minn, O. Yoon, and D. Chi. " On the First Eigenvalue Estimate of the Dirichlet and Neumann Problem," *Bull. Korean Math. Soc.* **23** (1968), 21-25.

[13] G. Weiss, " Partial Regularity for Weak Solutions of an Elliptic Free Boundary Problem, " *Comm. P. D. E.* **23** (1998), 439-457.



Wake Forest University

*E-mail address*: raynorsg@wfu.edu