Math 112
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TEST #3

NAME: Solutions

Pledge: I pledge on my honor that I have neither given nor received any assistance on this exam nor have I used any dishonest means to obtain my results.

Signature: ____________________________

Note: This test is out of 60 points. To receive full credit you must SHOW ALL WORK!

Some Formulae You May Find Useful:

$$\sin(\frac{\pi}{6}) = \frac{1}{2}$$

$$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Question	Score Possible	Score
1	30	
2	8	
3	10	
4	12	

Total Score: ______/ 60

1. (5 points each) For each of the following integrals, tell me what technique you need to use to solve the integral. **Your choices are: u-substitution, integration by parts, trig identity, trig substitution, or partial fractions.**

- If it is a u-substitution, state u and du and the changed integral in terms of u.
- If it is an integration by parts, state u, v, du, and dv, and write the integration by parts formula out.
- If it is a trig identity, state which one you would use and why.
- If it is a trig substitution, state the trig substitution you would use and give the new integral in terms of θ.
- If it is an integration by partial fractions, state the form that the partial fraction decomposition would take (but do not compute the constants).

IN NO CASE DO YOU NEED TO COMPUTE THE VALUE OF THE FINAL INTEGRAL.

Finally, some of these integrals may require a second technique after you complete the first one. If so, state what the second technique would be, but do not attempt to execute it.

(a) $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$ Integration by partial fractions

$x^3+4x = x(x^2+4)$

So, $\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

After that, we'll need to do a u-sub for the term $\frac{Bx}{x^2+4}$ ($u = x^2+4$ $du = 2x\,dx$) to finish the integral.

(b) $\int x^2\sqrt{25 - x^2}\,dx$ Trig substitution

$x = 5\sin\theta$ $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta}$

$dx = 5\cos\theta\,d\theta$ $= \sqrt{25\cos^2\theta} = 5\cos\theta$

$= \int 25\sin^2\theta\,(5\cos\theta)(5\cos\theta\,d\theta)$

$= 625\int \sin^2\theta\cos^2\theta\,d\theta$ We'll need to use the double angle formulae $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ to finish the integral.

(c) $\int \cos^3 x \sin^2 x \, dx$ u-sub: $u = \sin x$

$du = \cos x \, dx$

$= \int \cos^2 x \sin^2 x \, (\cos x \, dx)$

$= \int (1-\sin^2 x) \sin^2 x \cos x \, dx$

$= \int (1-u^2) u^2 du$ → This is a simple integral, so we don't need a second technique.

(d) $\int \frac{dx}{x\sqrt{1-[\ln(x)]^2}}$ u-sub: $u = \ln x$

$du = \frac{dx}{x}$

$= \int \frac{du}{\sqrt{1-u^2}}$ This integral is $\sin^{-1}(u)+C$ so we don't need a second technique.

(e) $\int \theta^3 \sec^2(\theta^2) d\theta$ u-sub: $u = \theta^2$

$du = 2\theta \, d\theta$

$= \frac{1}{2} \int \theta^2 \sec^2(\theta^2) \, (2\theta \, d\theta)$

$= \frac{1}{2} \int u \sec^2(u) du$ To finish the integral, we'll need to integrate by parts,

$u = u$ $dv = \sec^2(u) du$

$du = du$ $v = \tan(u)$

(f) $\int \sin^{-1}(x)dx$

integration by parts

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \frac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$= x\sin^{-1}(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$ to finish the integral, we'll need to do the u-sub $u = 1-x^2$

$du = -2x\,dx$

2. (8 points) Find the value of the integral or determine that it diverges: $\int_0^2 \frac{\ln(x)}{\sqrt{x}}dx$.

This integral is improper at 0,

$$\int_0^2 \frac{\ln(x)}{\sqrt{x}}dx = \lim_{t\to 0^+} \int_t^2 \frac{\ln(x)}{\sqrt{x}}dx$$

Integrate by Parts:

$u = \ln(x)$ $\quad dv = \frac{1}{\sqrt{x}}dx$

$du = \frac{dx}{x}$ $\quad v = 2\sqrt{x}$

$$= \lim_{t\to 0^+}\left(2\sqrt{x}\ln x - \int \frac{2\sqrt{x}}{\sqrt{x}\,\cancel{x}}\,dx\right)\Bigg|_t^2 \qquad \int \frac{2}{\sqrt{x}}dx = 4\sqrt{x} + C$$

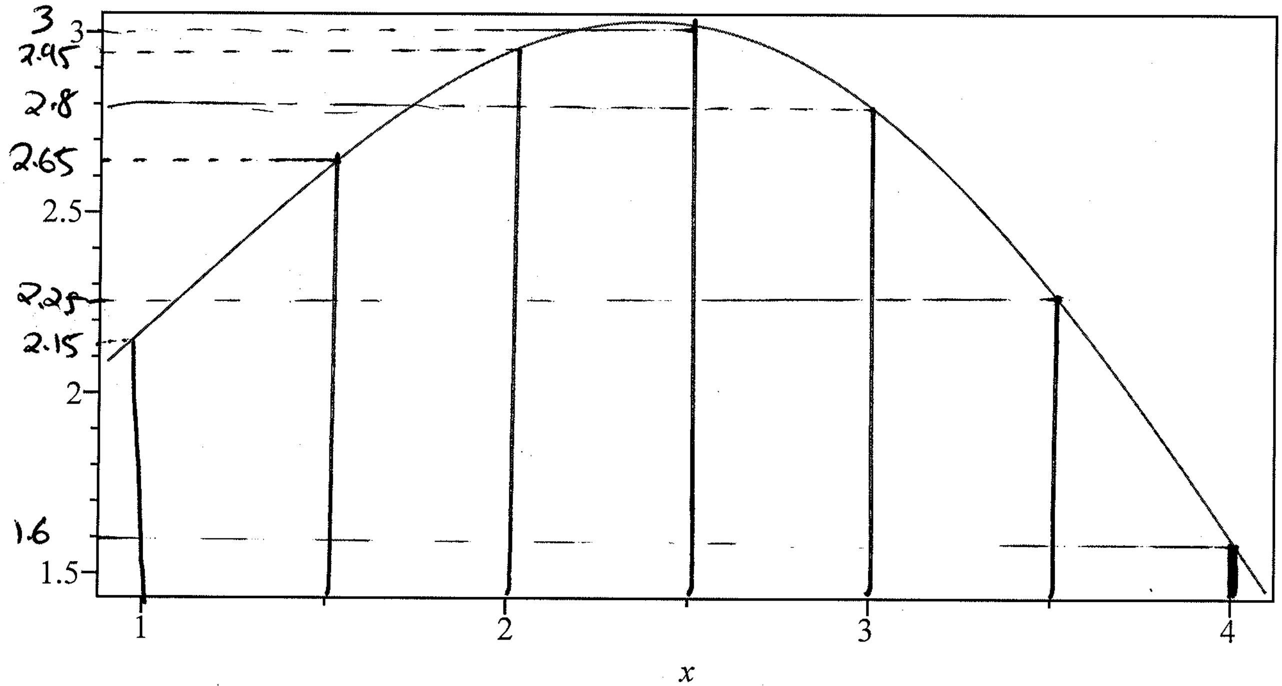
$$= \lim_{t\to 0^+}\left(2\sqrt{x}\ln x - 4\sqrt{x}\right)\Big|_t^2 = \lim_{t\to 0^+}\left(2\sqrt{2}\ln 2 - 4\sqrt{2} - 2\sqrt{t}\ln t + 4\sqrt{t}\right)$$

$\lim_{t\to 0^+} 4\sqrt{t} = 0$ What about $\lim_{t\to 0^+} \sqrt{t}\ln t$? $= 0\cdot(-\infty)$ indeterminate

$$= \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{\sqrt{t}}} = \frac{-\infty}{\infty} \quad \text{L'Hopital}$$

$$= \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}} = \lim_{t\to 0^+} \frac{-2t^{3/2}}{t} = \lim_{t\to 0^+} -2\sqrt{t} = 0.$$

So, $\lim_{t\to 0^+}\left(2\sqrt{2}\ln 2 - 4\sqrt{2} - 2\sqrt{t}\ln t + 4\sqrt{t}\right) = \boxed{2\sqrt{2}\ln 2 - 4\sqrt{2}}$



So,

x	f(x)
1	2.15
1.5	2.65
2	2.95
2.5	3
3	2.8
3.5	2.25
4	1.6

3. Consider the attached plot of a function $f(x)$.

(a) (5 points) Use the trapezoid Rule to estimate the value of $\int_1^4 f(x)dx$ with 6 subintervals. Will your answer be too large or two small and why?

$b-a=3$

$\Delta x = \frac{b-a}{n} = \frac{3}{6} = \frac{1}{2}$

$$\text{Trap} = \frac{\Delta x}{2}\big(f(1)+2f(1.5)+2f(2)+2f(2.5)+2f(3)+2f(3.5)+f(4)\big)$$

See graph page for approximate values of f

$$\approx \frac{1}{4}(2.15+5.3+5.9+6+5.6+4.5+1.6)$$

$$= \frac{1}{4}(31.05) = \boxed{7.7625} \quad \text{(ok to round more)}$$

This answer is too small because f is concave down on the interval.

(b) (5 points) Use Simpson's Rule to estimate the value of $\int_1^4 f(x)dx$ with 6 subintervals. Will your answer be more or less accurate than your answer from (a) and why?

$$\text{Simp} = \frac{\Delta x}{3}\big(f(1)+4f(1.5)+2f(2)+4f(2.5)+2f(3)+4f(3.5)+f(4)\big)$$

$$= \frac{1}{6}(2.15+10.6+5.9+12+5.6+9+1.6)$$

$$= \frac{1}{6}(46.85) = \boxed{7.8083}$$

This answer is more accurate because Simpson's Rule is far more accurate than the trapezoid rule.

4. Consider the sequence $a_n = 3(-\frac{2}{5})^n$, for $n = 0, 1, 2, \ldots$.

(a) (3 points) Write out the first 5 terms of a_n.

$a_0 = 3$ $\quad a_3 = \frac{-24}{125}$

$a_1 = -\frac{6}{5}$ $\quad a_4 = \frac{48}{625}$

$a_2 = \frac{12}{25}$

(b) (5 points) What type of sequence is (a_n)? Does it converge or diverge? If it converges, find the limit.

a_n is a geometric sequence. It equals $a \cdot r^n$ with $a=3$, $r = \frac{-2}{5}$. It converges to 0 because $|r|<1$ ($|r| = 2/5$) and any geometric sequence with $|r|<1$ converges to 0.

(c) (4 points) Consider the series $\sum_{n=0}^{\infty} a_n$. Does this series converge or diverge? If it converges, find the limit.

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 3(\frac{-2}{5})^n$. Since $|r|<1$, the series <u>converges</u> to $\frac{a}{1-r} = \frac{3}{1-\frac{-2}{5}} = \frac{3}{\frac{7}{5}} = \boxed{\frac{15}{7}}$