Math 112
Dr. Sarah Raynor
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TEST #2

NAME: Solutions

Pledge: I pledge on my honor that I have neither given nor received any assistance on this exam nor have I used any dishonest means to obtain my results.

Signature: ______________________

Note: This test is out of 72 points. To receive full credit you must SHOW ALL WORK!

Some Formulae You May find useful:

$$\sin(\frac{\pi}{6}) = \frac{1}{2}$$

$$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

Question	Score Possible	Score
1	30	
2	18	
3	7	
4	17	

Total Score: ______/ 72

1. Compute the following (5 points each)

(a) x if $\ln(x) = 2\ln(4) - \ln(x-6)$.

$$\ln(x) + \ln(x-6) = \ln(16)$$
$$\ln(x(x-6)) = \ln(16)$$
$$\ln(x^2 - 6x) = \ln(16)$$
$$x^2 - 6x = 16$$
$$x^2 - 6x - 16 = 0$$
$$(x-8)(x+2) = 0$$

$\rightarrow x = 8$ or -2

except when $x = -2$ $\ln(x)$ and $\ln(x-6)$ don't exist. So,

$$\boxed{x = 8}$$

(b) $\frac{d}{dx}(2^{x^2})$.

$$= 2^{x^2} \cdot \ln 2 \cdot 2x \qquad \text{(Chain rule)}$$
$$= \boxed{2\ln 2 \cdot x \cdot 2^{x^2}}$$

(c) $\int \tan^{-1}(x)dx$.

$u = \tan^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = x$

$$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx$$

$u = 1 + x^2$

$du = 2x\,dx$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{du}{u}$$
$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$$
$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C}$$

(d) $\lim_{x\to 0} \frac{\sin(x)}{\sin^{-1}(x)}$. $= \frac{0}{0}$ indeterminate

$$\text{L'Hop} = \lim_{x\to 0} \frac{\cos x}{\frac{1}{\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0} \cos(x)\sqrt{1-x^2}$$

$$= 1\cdot\sqrt{1-0} = \boxed{1}$$

(e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^2(x)}{\tan(x)}dx$

$u = \tan(x)$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{6} \to u = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{4} \to u = 1$

$$\int_{\frac{1}{\sqrt{3}}}^{1} \frac{du}{u} = \ln u \Big|_{\frac{1}{\sqrt{3}}}^{1} = \ln 1 - \ln\frac{1}{\sqrt{3}} = 0 - -\frac{1}{2}\ln 3 = \boxed{\frac{\ln 3}{2}}$$

(f) $\lim_{x\to\infty} x^{\frac{1}{x}}$ $= \infty^0$ indeterminate

Take the log: $\lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}$ $= \frac{\infty}{\infty}$ indet.

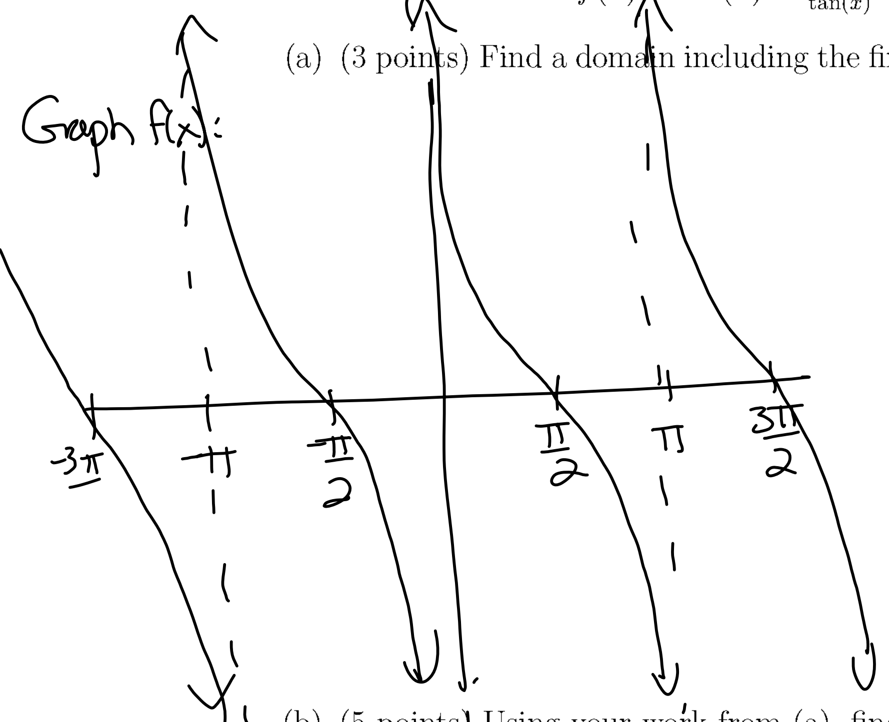
$$\text{L'Hop} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

So, $\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x} = e^0 = \boxed{1}$

2. Consider the function $f(x) = \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

(a) (3 points) Find a domain including the first quadrant on which $f(x)$ is one-to-one.

Graph f(x):



So a domain including the first quadrant where it's one-to-one is $\boxed{(0, \pi)}$

Notice the range on this domain is all reals.

(b) (5 points) Using your work from (a), find the domain and range of $f^{-1}(x)$.

The domain and range of $f^{-1}(x)$ are the range and domain of f, respectively. According to (a),

$\boxed{\text{Domain of } f^{-1} = \text{all reals} \quad \text{Range of } f^{-1} = (0, \pi)}$

(c) (2 points) Find $f^{-1}(\frac{-1}{\sqrt{3}})$.

$y = f^{-1}\left(\frac{-1}{\sqrt{3}}\right) \iff \frac{-1}{\sqrt{3}} = f(y)$

$\frac{-1}{\sqrt{3}} = \cot(y)$

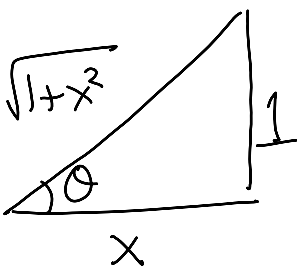
$\frac{\cos(y)}{\sin(y)} = \frac{-1}{\sqrt{3}}$

$y = 60°$ except it's negative, so $y = 120°$, i.e. $\boxed{\frac{2\pi}{3}}$

(d) (8 points) Find a formula for the derivative of $f^{-1}(x)$ **THAT DOES NOT HAVE ANY TRIG FUNCTIONS IN IT**.

$$f'(x) = -\csc^2(x)$$

So, $(f^{-1})'(x) = \dfrac{1}{-\csc^2(f^{-1}(x))} = \dfrac{-1}{\csc^2(\cot^{-1}(x))}$



$\theta = \cot^{-1}(x)$

$x = \cot(\theta)$

$= \dfrac{A}{O}$

So $H = \sqrt{1+x^2}$ by the pythagorean thm

So $\csc(\theta) = \dfrac{H}{O} = \dfrac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$

So, $(f^{-1})'(x) = \dfrac{-1}{(\sqrt{1+x^2})^2} = \boxed{\dfrac{-1}{1+x^2}}$

3. (7 points) State the formula for integration by parts. Explain in **FULL SENTENCES** what everything in the formula means, where it comes from, and how to use it.

The integration by parts formula is $\int u\,dv = uv - \int v\,du$ or $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$, where $u = f(x)$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$. The formula comes from the product rule $(fg)' = fg' + gf'$. Integrating the whole equation and solving for the term $\int f(x)g'(x)dx$ gives the equation above. You use it to integrate functions that are products with no composed functions involved. Choose f, or u, to be the portion of the integral that will get easier when you differentiate it, and dv to be the rest of the integrand, which shouldn't get harder when integrated.

4. You have taken out a loan for \$100,000 at a 9% annual interest rate. You intend to pay \$1000 per month to pay off this loan. This situation can be modelled by a function M(t) which solves the equation

$$\frac{dM(t)}{dt} = 0.0075M(t) - 1000,$$

where t is in units of months.

(a) (7 points) Find an expression for $M(t)$, the amount of money you owe as a function of time. (Hint: Let $y = M(t) - 1000/0.0075 = M(t) - 400000/3$. Then $y(t)$ should be a familiar function.)

$\frac{dy}{dt} = \frac{dM}{dt}$, so $\frac{dy}{dt} = .0075\left(M - \frac{1000}{.0075}\right)$

$= .0075y$

So, $y(t) = y(0)e^{.0075t}$

and $M(t) = y(0)e^{.0075t} + \frac{400,000}{3}$

At $t=0$, $M(0) = 100,000 = y(0)e^{0} + \frac{400,000}{3}$

So, $y(0) = \frac{-100,000}{3}$

And thus $\boxed{M(t) = \frac{-100,000}{3}e^{.0075t} + \frac{400,000}{3}}$

(b) (5 points) When will your loan be paid off? That is, how long will it be until you don't owe any more money? You should give an exact answer, not a decimal.

Solve for when $M(t) = 0$

$0 = \frac{-100,000}{3}e^{.0075t} + \frac{400,000}{3}$

$\frac{4\cancel{00,000}}{\cancel{3}} = \frac{1\cancel{00,000}}{\cancel{3}}e^{.0075t}$

$4 = e^{.0075t}$

$\rightarrow .0075t = \ln 4$

$t = \frac{400}{3}\ln 4$

$\boxed{t = \frac{800}{3}\ln 2 \text{ months}}$

$\boxed{\text{(In fact, } t \sim 15.4 \text{ yrs.)}}$

(c) (5 points) Sketch a graph of the function $M(t)$ on the attached graph paper, including any asymptotes and intercepts. You don't need to try to find critical points or inflection points. (There aren't any.)