Math 112
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February 11, 2009

TEST #1

NAME: Solutions

Pledge: I pledge on my honor that I have neither given nor received any assistance on this exam nor have I used any dishonest means to obtain my results.

Signature: ____________________

Note: This test is out of 65 points. To receive full credit you must SHOW ALL WORK!

Some Formulae You May find useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sin(\frac{\pi}{4}) = \sin(\frac{3\pi}{4}) = \cos(\frac{\pi}{4}) = -\cos(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$$

Question	Score Possible	Score
1	15	
2	5	
3	10	
4	10	
5	17	
6	8	

Total Score: ______/ 65

1. Compute the following: (5 points each)

(a)

$$\int_1^2 \frac{x^3 - \sqrt[4]{x}}{x} dx$$

$$= \int_1^2 (x^2 - x^{-3/4}) dx$$

$$= \left(\frac{x^3}{3} - 4x^{1/4}\right)\Big|_1^2 = \frac{8}{3} - 4\sqrt[4]{2} - \frac{1}{3} + \overset{\frac{12}{3}\ ||}{4}$$

$$= \boxed{\frac{19}{3} - 4\sqrt[4]{2}}$$

(b)

$$\int \frac{\sin\left(\frac{\pi}{x}\right)}{x^2} dx = \frac{-1}{\pi}\int \sin\left(\frac{\pi}{x}\right) \frac{-\pi}{x^2} dx$$

$u = \frac{\pi}{x}$

$du = \frac{-\pi}{x^2} dx$

$$= \frac{-1}{\pi}\int \sin(u) du$$

$$= \frac{-1}{\pi}(-\cos(u)) + C$$

$$= \boxed{\frac{1}{\pi}\cos\left(\frac{\pi}{x}\right) + C}$$

(c) The average value of the function $f(x) = \frac{2x}{(1+x^2)^2}$ over the interval $[0, 2]$.

length of interval = 2

$$f_{ave} = \frac{1}{\cancel{2}}\int_0^2 \frac{\cancel{2}x}{(1+x^2)^2} dx$$

$u = 1 + x^2$

$du = 2x dx$

$x = 0 \to u = 1$

$x = 2 \to u = 5$

$$= \frac{1}{2}\int_1^5 \frac{1}{u^2} du$$

$$= \frac{-1}{2u}\Big|_1^5 = \frac{-1}{10} + \frac{1}{2} = \frac{-1+5}{10} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

4. (10 points) Consider the function $f(x) = x^2$. Use the **definition** of the definite integral to find the area under f from $x = 1$ to $x = 3$.

$a = 1$

$b = 3$

$b - a = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

$$x_i = a + \frac{b-a}{n} i = 1 + \frac{2i}{n}$$

$$f(x_i) = \left(1 + \frac{2i}{n}\right)^2 = 1 + \frac{4i}{n} + \frac{4i^2}{n^2}$$

$$f(x_i)\Delta x = \frac{2}{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) = \frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3}$$

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{2}{\cancel{n}}\cancel{n} + \frac{\cancel{8}^{\,4}}{n^2}\frac{n(n+1)}{\cancel{2}} + \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= 2 + 4\left(\frac{n(n+1)}{n^2}\right) + \frac{4}{3}\,\frac{n(n+1)(2n+1)}{n^3}$$

$$\text{Exact area} = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(2 + 4\underset{\substack{\downarrow \\ \text{limit is 1}}}{\frac{n(n+1)}{n^2}} + \frac{4}{3}\underset{\substack{\downarrow \\ \text{limit is 2}}}{\frac{n(n+1)(2n+1)}{n^3}}\right)$$

$$= 2 + 4 + \frac{4}{3}\cdot 2$$

$$= 6 + \frac{8}{3} = \boxed{\frac{26}{3}}$$

Check my answer: $\displaystyle\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$ ✓

2. (5 points) If the velocity of a particle is given by $v(t) = \sec^2(t)$, and the position of the particle at $t = \frac{\pi}{4}$ is known to be $x(\frac{\pi}{4}) = 3$, then find the position of the particle $x(t)$ as a function of time for all values of t.

$x(t) = \tan(t) + C$

$x(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) + C = 3$ $\qquad \tan(\frac{\pi}{4}) = 1$

$1 + C = 3$

$C = 2$

$\boxed{x(t) = \tan(t) + 2}$

3. Consider the function

$$F(x) = \int_{\pi}^{x} \frac{t+1}{t^2+3} dx$$

(a) (5pts) Find the values of x for which F has a maximum or minimum.

max or min → $F'(x) = 0$ (critical point)

By FTC 1, $F'(x) = \frac{x+1}{x^2+3} = 0$ when $x = -1$

Max or min?

First derivative test: $\xleftarrow{\quad - \quad} \underset{-1}{|} \xrightarrow{\quad + \quad}$ ⇒ $\boxed{-1 \text{ is a minimum of } F}$

(b) (5 pts) Find the points of inflection of F.

IPs: $F''(x) = 0$

$F''(x) = (F'(x))' = \left(\frac{x+1}{x^2+3}\right)' = \frac{x^2+3-(x+1)(2x)}{(x^2+3)^2}$

$= \frac{x^2+3-2x^2-2x}{(x^2+3)^2} = \frac{-x^2-2x+3}{(x^2+3)^2} = \frac{-(x+3)(x-1}{(x^2+3)^2}$

$= 0$ when $x = \boxed{-3, 1}$

5. (a) (5 pts) Find the area between the functions $x = 2y^2$ and $y = -\frac{1}{2}x + 2$.

↓ solve for x

$y-2=-\frac{1}{2}x$

$x=-2y+4$

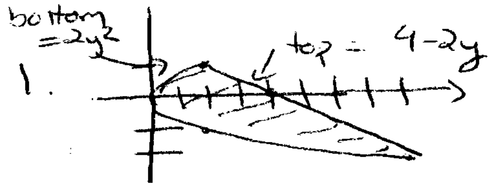
Intersection points.

$2y^2=4-2y$

$2y^2+2y-4=0$

$2(y^2+y-2)=0$

$2(y+2)(y-1)=0$ $\quad y=-2, 1.$



I want to integrate dy.

$$\int_{-2}^{1}(4-2y-2y^2)\,dy=\left(4y-y^2-\frac{2}{3}y^3\right)\Big|_{-2}^{1}=4-1-\frac{2}{3}-\left(-8-4+\frac{16}{3}\right)$$

$$=15-\frac{18}{3}=\boxed{9}$$

(b) (7 points) Find the volume generated by rotating the area in part (a) around the y-axis.

dy ⇒ use washers.

$$V=\pi\int_{-2}^{1}\left((4-2y)^2-(2y^2)^2\right)dy$$

$$=\pi\int_{-2}^{1}(16-16y+4y^2-4y^4)\,dy$$

$$=\pi\left(16y-8y^2+\frac{4}{3}y^3-\frac{4}{5}y^5\right)\Big|_{-2}^{1}$$

$$=\pi\left(16-8+\frac{4}{3}-\frac{4}{5}\right)-\pi\left(-32-32-\frac{32}{3}+\frac{128}{5}\right)$$

$$=\pi\left(72+\overset{12}{\cancel{\frac{36}{3}}}-\frac{132}{5}\right)$$

$$=\boxed{\frac{288\pi}{5}}$$

$\frac{132}{5}=26\,{}^{2}/_{5}$

$84-26=58$

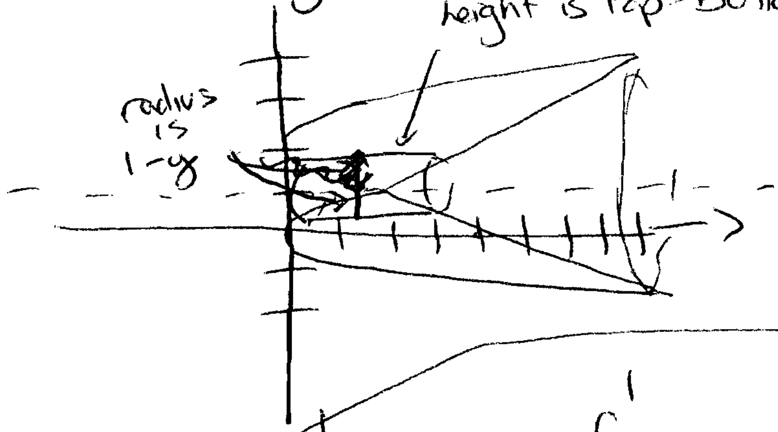
$57\,{}^{3}/_{5}=\frac{288}{5}$

$$\begin{array}{r} {}^{3} \\ 57 \\ \times 5 \\ \hline 285 \end{array}$$

(c) (5 points) Set up but do not evaluate an integral that equals the volume generated by rotating the area in part (a) around the line $y = 1$.

dy → Shells

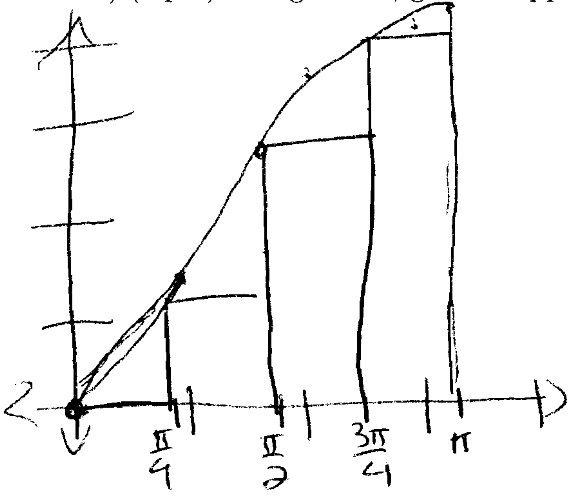
height is top-bottom = $4-2y-2y^2$



$$\boxed{V = 2\pi\int_{-2}^{1}(1-y)(4-2y-2y^2)\,dy}$$

6. Consider the function $f(x) = x + \sin(x)$.

a) (5 pts) Using $n = 4$, give an approximation of the net area under f from 0 to π.



Use left method.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$A \cong \frac{\pi}{4}\left(f(0) + f(\tfrac{\pi}{4}) + f(\tfrac{\pi}{2}) + f(\tfrac{3\pi}{4}\right.$$

$$= \frac{\pi}{4}\left(0 + \frac{\pi}{4} + \frac{\sqrt{2}}{2} + \frac{\pi}{2} + 1 + \frac{3\pi}{4} + \frac{\sqrt{2}}{2}\right)$$

$$= \boxed{\frac{\pi}{4}\left(\frac{3\pi}{2} + \sqrt{2} + 1\right)}$$

x	f(x)
0	$0+0=0$
$\frac{\pi}{4}$	$\frac{\pi}{4}+\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	$\frac{\pi}{2}+1$
$\frac{3\pi}{4}$	$\frac{3\pi}{4}+\frac{\sqrt{2}}{2}$

b) (3 pts) Is your answer too small or too large? Justify your response.

My answer is too small, because this function is increasing on the interval, ($f'(x) = 1 + \cos(x) \geq 0$ for all x)

We know that the left hand method is always too small for all increasing functions. We can also see this from the picture, because all of the rectangles are always below the function.