Math 112
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TEST #2

NAME: Solutions

Pledge: I pledge on my honor that I have neither given nor received any assistance on this exam nor have I used any dishonest means to obtain my results.

Signature: ____________________

Note: This test is out of 50 points. To receive full credit you must SHOW ALL WORK!

Some Formulae You May find useful:

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Question	Score Possible	Score
1	29	
2	6	
3	10	
4	5	

Total Score: ______/ 50

1. Compute the following:

(a) (7 points)

$$\int \frac{x^2}{x^2+5x+4}dx$$

$$x^2+5x+4=(x+1)(x+4)$$

$$\frac{x^2}{x^2+5x+4}=\frac{x^2+5x+4-5x-4}{x^2+5x+4}$$

$$=1-\frac{5x+4}{x^2+5x+4}$$

$$\frac{5x+4}{x^2+5x+4}=\frac{A}{x+1}+\frac{B}{x+4}$$

$$5x+4=A(x+4)+B(x+1)$$

$$=(A+B)x+4A+B$$

$A+B=5$ $\quad 4A+B=4$

$3A=-1$ $\quad B=\frac{16}{3}$

$A=-1/3$

$$=\int\left(1-\frac{1}{3(x+1)}+\frac{16}{3(x+4)}\right)dx=\boxed{x-\frac{1}{3}\ln|x+1|+\frac{16}{3}\ln|x+4|+C}$$

(b) (7 points)

$$\int_1^2 \ln(x)^2dx$$

Integrate by Parts:

$u=(\ln(x))^2$ $\quad dv=dx$

$du=\frac{2\ln(x)}{x}dx$ $\quad v=x$

$$=x(\ln(x))^2-\int\frac{2\ln x}{\cancel{x}}\cancel{x}\,dx$$

Again:

$u=\ln x$ $\quad dv=dx$

$du=\frac{dx}{x}$ $\quad v=x$

$$=x(\ln(x))^2-2\left(x\ln x-\int \cancel{x}\frac{dx}{\cancel{x}}\right)$$

$$=\left(x(\ln(x))^2-2x\ln x+2x\right)\Big|_1^2$$

$$=2(\ln 2)^2-4\ln 2+4-2$$

$$=\boxed{2\ln(2)(\ln(2)-2)+2}$$

(c) (5 points) $\lim_{x\to\frac{\pi}{2}}(\sec(x)-\tan(x))$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos(x)}-\frac{\sin(x)}{\cos(x)}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin(x)}{\cos(x)}=\frac{0}{0}$$

L'Hop $$=\lim_{x\to\frac{\pi}{2}}\frac{-\cos(x)}{-\sin(x)}=\frac{-0}{-1}=\boxed{0}$$

(d) (5 points)

$$\int\frac{\cos(\theta)}{1+\sin^2(\theta)}d\theta$$

$u=\sin\theta$
$du=\cos\theta\, d\theta$

$$=\int\frac{du}{1+u^2}=\tan^{-1}(u)+C$$

$$=\boxed{\tan^{-1}(\sin(\theta))+C}$$

(e) (5 points)

$$\frac{d}{dx}\ln(\tan^{-1}(x)).$$

$$=\frac{1}{\tan^{-1}(x)}\cdot\frac{1}{1+x^2}$$ Chain rule

$$=\boxed{\frac{1}{(1+x^2)\tan^{-1}(x)}}$$

2. (6 points) For each function in the list below, match it to one of the attached graphs. Justify your choices using full sentences. Note that the constants C and a are unknown positive numbers, but they are the same for all the graphs. Since you don't know what C and a are, your answer should be based on the qualitative shapes of the graphs.

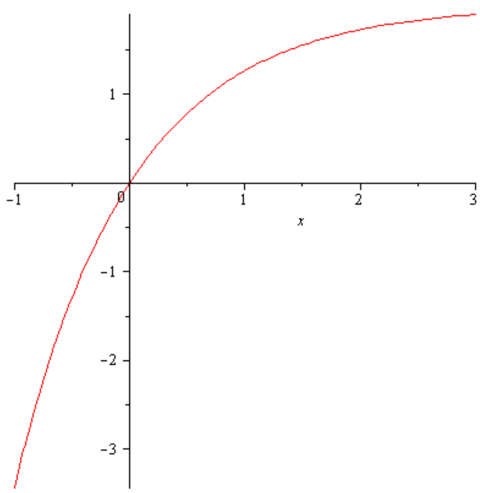
(a) Ce^{ax} This is $\overline{\text{III}}$. We know that Ce^{ax} will have an asymptote at 0 as $x \to -\infty$ and will go to ∞ quickly as $x \to +\infty$. III is the only one meeting those rules.

(b) $C(1-e^{-ax})$ This is I. It should have an HA at C as $x \to +\infty$ and go to $-\infty$ quickly as $x \to -\infty$, just like I does. (It looks like C is about 2.)

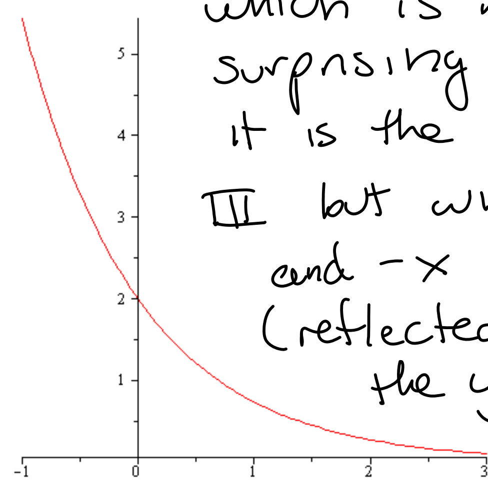
(c) $\frac{C}{1+e^{-ax}}$. This is $\overline{\text{IV}}$. This function is the only one with two HAs, at 0 as $x \to -\infty$, and at C as $x \to +\infty$.

(d) Ce^{-ax} By process of elimination, this is II, which is not surprising because it is the same as III but with x and -x switched (reflected across the y-axis).

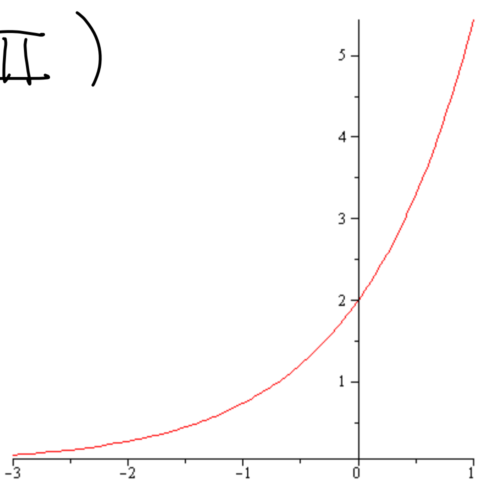
(I)



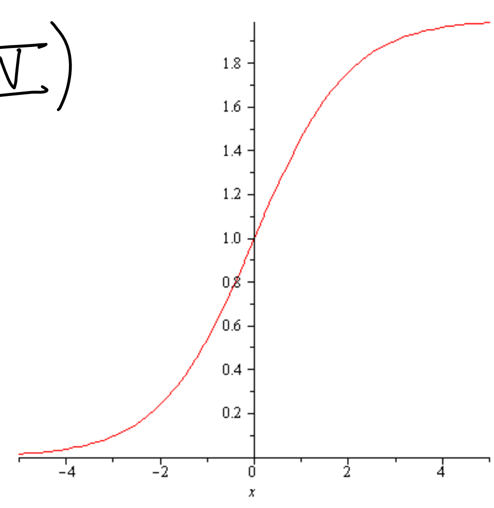
(II)



(III)



(IV)



3. (10 points) The value of a car declines exponentially after its date of purchase. Suppose you initially purchased a car for \$12,000. After 1 year, it is worth only \$9,000. Find a function which models the value of your car over time, and use that function to find out how long it will take until your car is only worth \$1,000. If you took a 5 year loan out to pay for the car, would it be paid off before your car became worthless? If so, how much would the car be worth when your loan was paid off?

Let $V(t)$ = the value of the car at time t, with t measured in years.

$$\frac{dV}{dt} = -\alpha V, \quad \text{so} \quad V(t) = V(0)e^{-\alpha t}$$

$$V(t) = 12{,}000\, e^{-\alpha t}$$

$$V(1) = 9000 = 12{,}000\, e^{-\alpha 1}$$

$$\frac{3}{4} = e^{-\alpha} \qquad -\alpha = \ln\frac{3}{4}$$

$$\alpha = \ln\left(\frac{4}{3}\right)$$

$$V(t) = 12{,}000 \cdot e^{-\ln\left(\frac{4}{3}\right)t} = 12{,}000\left(\frac{3}{4}\right)^t$$

When is the value \$1000?

$$1000 = 12{,}000\left(\frac{3}{4}\right)^t$$

$$\frac{1}{12} = \left(\frac{3}{4}\right)^t \rightarrow -\ln 12 = -t\ln\left(\frac{4}{3}\right)$$

$$\boxed{t = \frac{\ln 4 + \ln 3}{\ln 4 - \ln 3}}$$

We know the car never becomes absolutely worthless, because an exponential function can never be exactly 0. So yes, it is paid off before it is worthless. In fact, after 5 years, the car is worth

$$V(5) = 12{,}000\left(\frac{3}{4}\right)^5 = \boxed{\$2847.66}$$

4. (5 points) Find the absolute maximum and minimum of the function

$$f(x) = \log_2(1 + x + x^2)$$

on the interval $[-1, 1]$.

$$f'(x) = \frac{1}{\ln 2} \frac{1}{1+x+x^2}(1+2x) = 0$$

The only CP occurs when $1+2x=0 \to x = \frac{-1}{2}$

We also need to check the endpoints:

x	$f(x)$
-1	$\log_2(1-1+1) = \log_2(1) = 0$
$-\frac{1}{2}$	$\log_2\left(1-\frac{1}{2}+\frac{1}{4}\right) = \log_2\left(\frac{3}{4}\right) = -\log_2\left(\frac{4}{3}\right) < 0$
1	$\log_2(1+1+1) = \log_2(3) > 0$

So, $\boxed{\text{the absolute max is } \log_2 3\text{, occurring at the endpoint } x=1\text{, while the absolute min is } -\log_2\left(\frac{4}{3}\right)\text{, occurring at the CP } x = \frac{-1}{2}.}$