Math 112
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Homework #9 Solutions

#1) $a_1 = \frac{1}{1}$ $a_2 = \frac{1}{1\cdot 3}$ $a_3 = \frac{1}{1\cdot 3\cdot 5}$ $a_4 = \frac{1}{1\cdot 3\cdot 5\cdot 7}$

So, $\boxed{a_n = \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}}$

#2) $a_1 = 2$ Suppose $0<a_n<2$. Then

$a_{n+1} = \frac{1}{3-a_n}$ $a_{n+1} < \frac{1}{3-2} = \frac{1}{1} = 1$

and $a_{n+1} > \frac{1}{3-0} = \frac{1}{3} = \frac{1}{3}$

Thus $0<a_{n+1}<2$ if $0<a_n<2$.

Since a_1 satisfies $0<a_1<2$, by induction $0<a_n<2$ for all n.

Now, $a_2 - a_1 = \frac{1}{3-2} - 2 = \frac{1}{1} - 2 = -1 < 0$.

Suppose we know that $a_n < a_{n-1}$. Then $a_{n+1} = \frac{1}{3-a_n} < \frac{1}{3-a_{n-1}} = a_n$.

Hence, by induction, (a_n) is a decreasing sequence because a_{n+1} is always less than a_n.

Because (a_n) is a bounded, decreasing sequence, by the Monotone Convergence Theorem, it converges. To find its limit, we know $a_{n+1} = \frac{1}{3-a_n}$, so if it converges

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{3-\lim_{n\to\infty} a_n}$$

$$A = \frac{1}{3-A} \quad \text{if} \quad A = \lim_{n\to\infty} a_n$$

$$A(3-A) = 1$$

$$3A - A^2 - 1 = 0$$

$$A^2 - 3A + 1 = 0 \qquad A = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2}$$

But we know $0<A<2$, so $\boxed{A = \frac{3-\sqrt{5}}{2}}$.

#3) $L = 2.5\overline{37} = 2 + \frac{5}{10} + \frac{3}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{7}{100000} + \cdots$

$= 2 + \frac{5}{10} + \frac{37}{1000} + \frac{37}{100000} + \cdots$

$$L = 2.5 + \frac{37}{1000}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$$

$$= 2.5 + \frac{37}{1000}\sum_{n=0}^{\infty}\frac{1}{100^n}$$ ← geometric series representation of L

$$= \frac{5}{2} + \frac{37}{1000}\left(\frac{1}{1-\frac{1}{100}}\right)$$ because $r = \frac{1}{100}$

$$= \frac{5}{2} + \frac{37}{1000}\left(\frac{100}{100-1}\right) = \frac{5}{2} + \frac{37}{10\cancel{00}} \cdot \frac{1\cancel{00}}{99}$$

$$= \frac{5}{2} + \frac{37}{990} = \frac{5 \cdot 495 + 37}{990} = \frac{2512}{990} = \boxed{\frac{1256}{495}}$$

#4) $\sum_{n=1}^{\infty}\frac{(x+1)^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{x+1}{3}\right)^n$ geometric, with $r = \frac{x+1}{3}$

Converges when $|r| < 1$, i.e. $\left|\frac{x+1}{3}\right| < 1$

$$|x+1| < 3$$

$$-3 < x+1 < 3$$

$$\boxed{-4 < x < 2}$$

When it converges, the limit is $\frac{a}{1-r} = \frac{\left(\frac{x+1}{3}\right)}{1-\left(\frac{x+1}{3}\right)}$

$$= \frac{x+1}{3-(x+1)} = \boxed{\frac{x+1}{4-x}}$$