Math 112
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Homework #8 Solutions

#1) $\displaystyle\int_0^2 \frac{\ln x}{x^p}\,dx$ p any real. Integrate by parts: (unless $p=1$)

$u = \ln x \qquad dv = x^{-p}dx$

$du = \frac{dx}{x} \qquad v = \frac{x^{1-p}}{1-p}$

$$= \lim_{s\to 0} \int_s^2 \frac{\ln x}{x^p}\,dx = \lim_{s\to 0}\left(\frac{x^{1-p}}{1-p}\ln x \Big|_s^2 - \int_s^2 \frac{x^{-p}}{1-p}\,dx \right)$$

$$= \lim_{s\to 0}\left(\frac{x^{1-p}}{1-p}\ln x \Big|_s^2 - \frac{x^{1-p}}{(1-p)^2}\Big|_s^2 \right)$$

$$= \lim_{s\to 0}\left(\frac{2^{1-p}}{1-p}\ln 2 - \frac{s^{1-p}}{1-p}\ln s - \frac{2^{1-p}}{(1-p)^2} + \frac{s^{1-p}}{(1-p)^2} \right)$$

If $p>1$, $\lim_{s\to 0} \frac{s^{1-p}}{1-p} = -\infty$ so this diverges.

If $p<1$, $\lim_{s\to 0} \frac{s^{1-p}}{1-p} = 0$. What about $\lim_{s\to 0} \frac{s^{1-p}}{1-p}\ln s = 0\cdot(-\infty)$ Indeterminate

$$= \lim_{s\to 0} \frac{1}{1-p}\frac{\ln s}{s^{p-1}} \overset{\text{L'Hop}}{=} \lim_{s\to 0} \frac{1}{(1-p)} \frac{\frac{1}{s}}{(p-1)s^{p-2}}$$

$$= \lim_{s\to 0} \frac{-1}{(1-p)^2} s^{1-p} = 0.$$ So the integral converges

What if $p=1$? $\displaystyle\int_0^2 \frac{\ln x}{x}\,dx = \lim_{s\to 0}\int_s^2 \frac{\ln x}{x}\,dx$ $\qquad u=\ln x$, $du = \frac{dx}{x}$

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{(\ln x)^2}{2}+C$$

$$= \lim_{s\to 0} \frac{\ln(x)^2}{2}\Big|_s^2$$

$$= \lim_{s\to 0}\left(\frac{(\ln(2))^2}{2} - \frac{(\ln(s))^2}{2} \right) = -\infty$$

Diverges

$\boxed{\text{So, this integral converges for all } p<1 \text{ and diverges for all } p\geq 1.}$

2) The comparison theorem says nothing about $\int_1^\infty \frac{|\sin(x)|}{x}dx$, because $0 \leq \frac{|\sin(x)|}{x} \leq \frac{1}{x}$ for all x, but $\int_1^\infty \frac{dx}{x}$ diverges, so this inequality goes the wrong way to say anything using the comparison theorem. I think this integral actually diverges, because $|\sin(x)|$ is greater than or equal to $\frac{1}{2}$ for all x from $n\pi + \frac{\pi}{6}$ to $n\pi + \frac{5\pi}{6}$, which means that actually $\frac{|\sin(x)|}{x} \geq \frac{1}{2x}$ for most x, and $\int^\infty \frac{dx}{2x}$ diverges, so probably $\int^\infty \frac{|\sin(x)|}{x}dx$ diverges too by the comparison th'm. I tested this by trying to do $\int_1^T \frac{|\sin(x)|}{x}dx$ on Maple for large values of T, and it crashed, which probably means it diverges.

The comparison test doesn't apply at all to $\int^\infty \frac{\sin x}{x}dx$ because the integrand is not always nonnegative. However, I tested it on Maple for $\int_1^T \frac{\sin x}{x}dx$ for T as large as 1,000,000,000 and it seems to converge to 0.624713. It probably converges due to the cancellation of positive and negative terms from $\sin(x)$.

#3) See Maple printout.

Problem #24 from Section 8.7

```
> restart:with(plots):
> f:=x->sqrt(4-x^3);
```

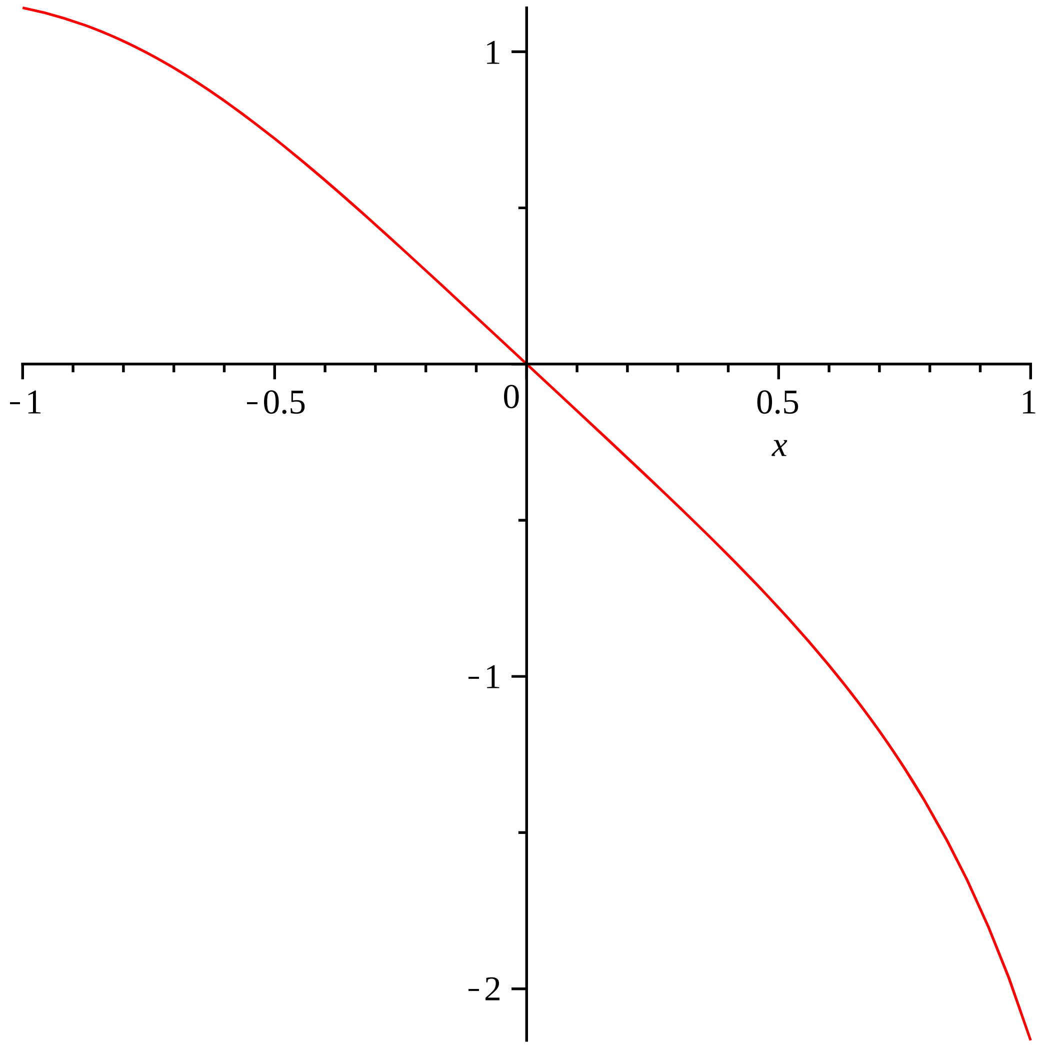
$$f := x \to \sqrt{4 - x^3} \quad (1)$$

```
> a:=-1;b:=1;
```

$$a := -1$$
$$b := 1 \quad (2)$$

Part (a)

```
> plot(diff(diff(f(x),x),x),x=a..b);
```



Based on the graph, f'' varies from a little below -2 to just above 1, so we can bound |f''| by 5/2.

Part (b)

```
> with(Student[Calculus1]):
  A:=evalf(ApproximateInt(f(x),x=a..b,method=midpoint));
```

$$A := 3.995804152 \quad (3)$$

Part (c)

According to the formula, E_M is controlled by max(|f''|)*(b-a)^3/(24*n^2), which is at most (2.5*8)/

(24*100)=1/120=.008333.

Part (d)

```
> B:=evalf(int(f(x),x=a..b));
```

$B := 3.995487699$ **(4)**

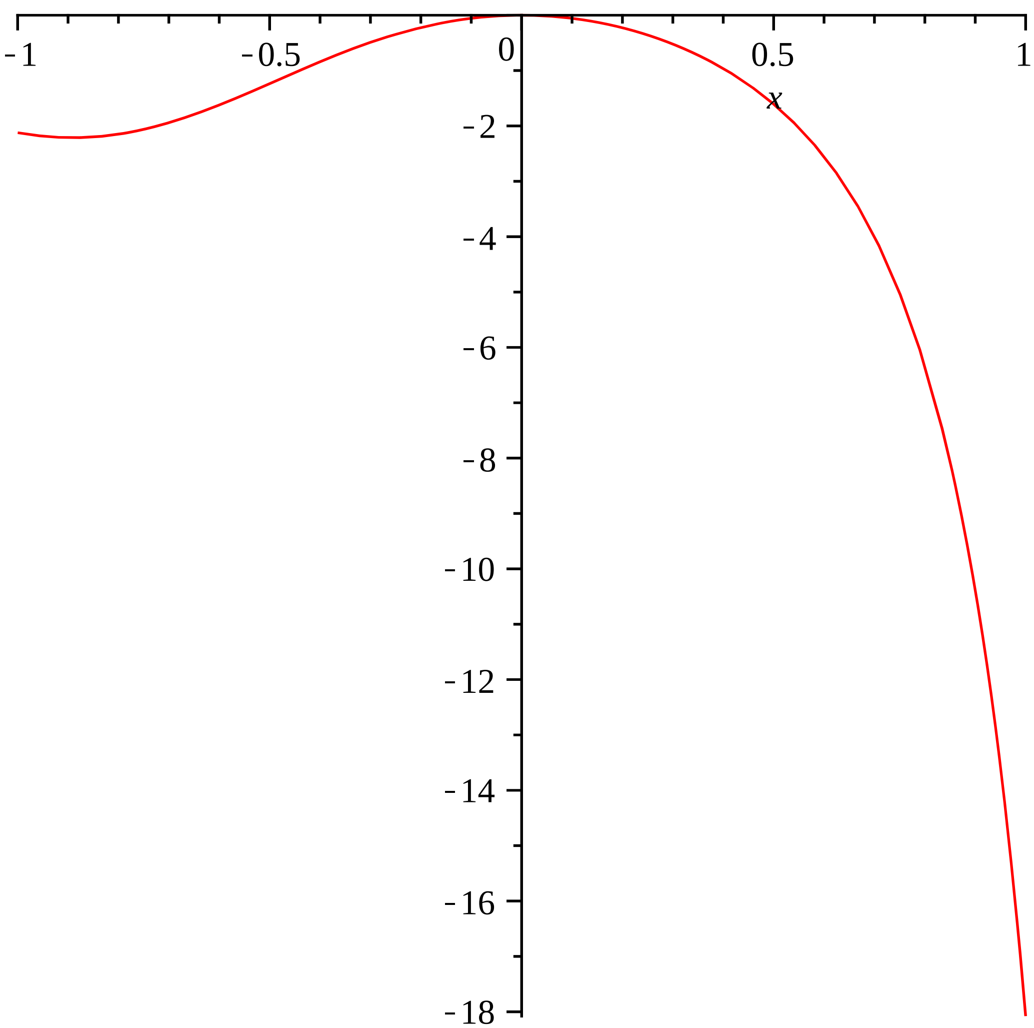
Part (e)

```
> A-B;
```

The actual error is much smaller than the upper bound of the error which we predicted in part (c).

Part (f)

```
> plot(diff(f(x),x,x,x,x),x=a..b);
```



Based on the picture, we can conclude that |f""(x)| can be bounded by 18.

Part (g)

```
> C:=evalf(ApproximateInt(f(x),x=a..b,method=simpson));
```

$C := 3.995485150$ **(5)**

Part (h)

According to the formula, E_S is controlled by max(|f""|)*(b-a)^5/(180*n^4), which is at most (18*32)/180*10000)=.00032.

```
> 18*32/(180*10000);
  evalf(%);
```

$$\frac{1}{3125}$$

$$0.0003200000000 \tag{6}$$

Part (i)

```
> C-B;
```

$$-0.000002549 \tag{7}$$

The actual error is much smaller than the upper bound of the error which we predicted in part (h).

Part (j)

We want to solve for 18*32/(180*n^4) < .0001, i..e. n^4 > 18*32*10000/180.

```
> (18*32*10000/180)^(1/4);
  evalf(%);
```

$$32000^{1/4}$$

$$13.37480610 \tag{8}$$

We need at least n=14 partitions to guarantee that the error is at most .0001.

#4) a) $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx$ Let $u = x^2$, $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{u^2+u+1}$$

Complete the square

$u^2+u+1 = u^2+u+\frac{1}{4}+\frac{3}{4} = \left(u+\frac{1}{2}\right)^2+\frac{3}{4}$

$$= \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2\cdot\frac{3}{4}}\int \frac{du}{\frac{4}{3}\left(u+\frac{1}{2}\right)^2+1} = \frac{2}{3}\int \frac{du}{1+\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2}$$

Let $v = \frac{2u}{\sqrt{3}} + \frac{1}{\sqrt{3}}$, $dv = \frac{2}{\sqrt{3}}\,du$

$$= \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{dv}{1+v^2} = \frac{1}{\sqrt{3}}\tan^{-1}(v)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+C = \boxed{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+C}$$

b) $\displaystyle\int \ln(x^2-1)\,dx$ $\ln(x^2-1) = \ln((x-1)(x+1)) = \ln(x-1)+\ln(x+1)$

$$= \int (\ln(x-1)+\ln(x+1))\,dx \qquad \int \ln x\,dx = x\ln x - x + C$$

$$= (x-1)\ln(x-1) - (x-\cancel{1}) + (x+1)\ln(x+1) - (x+\cancel{1}) + C$$

$$= \boxed{x\ln(x^2-1) + \ln\left(\frac{x+1}{x-1}\right) - 2x + C}$$

c) $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}} = \int \frac{dx}{\sqrt{x}(1+x)}$ $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$ $1+x = 1+(\sqrt{x})^2 = 1+u^2$

$$= 2\int \frac{du}{1+u^2} = 2\tan^{-1}(u)+C = \boxed{2\tan^{-1}(\sqrt{x})+C}$$