Math 112
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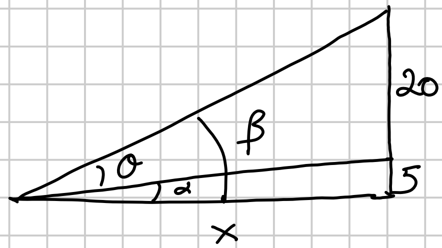
Fall 2009

Homework #7 Solutions

#1)



Maximize θ as a function of x:

$$\theta = \beta - \alpha$$

$$\tan\beta = \frac{25}{x} \qquad \tan\alpha = \frac{5}{x}$$

$$\beta = \tan^{-1}\left(\frac{25}{x}\right) \qquad \alpha = \tan^{-1}\left(\frac{5}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{25}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right)$$

$$0 = \frac{d\theta}{dx} = \frac{1}{1+\left(\frac{25}{x}\right)^2}\cdot\frac{-25}{x^2} - \frac{1}{1+\left(\frac{5}{x}\right)^2}\cdot\frac{-5}{x^2}$$

$$= \frac{-25}{x^2+25^2} + \frac{5}{x^2+5^2}$$

$$= \frac{-25(x^2+25)+5(x^2+625)}{(x^2+625)(x^2+25)}$$

Denominator is always positive

So,

$$0 = -25x^2 - 625 + 5x^2 + 3125$$

$$20x^2 = 2500$$

$$x^2 = 125$$

$x = 5\sqrt{5}$. You should sit about $\boxed{11.18\text{ ft}}$ from the base of the screen.

#2) $\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$

Consider $\lim_{x\to\infty} \ln\left((e^x+x)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{1}{x}\ln(e^x+x)$

$$= \lim_{x\to\infty} \frac{\ln(e^x+x)}{x} = \frac{\infty}{\infty}$$

indeterminate

Do L'Hopital's Rule:
$$= \lim_{x\to\infty} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} = \frac{\infty}{\infty} \text{ L'Hopital}$$

L'Hopital Again
$$= \lim_{x\to\infty} \frac{e^x}{e^x+1}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{e^x}} = \lim_{x\to\infty} \frac{1}{1+e^{-x}}$$

$$= 1$$

So $\lim_{x\to\infty} (e^x+x)^{\frac{1}{x}} = e^{\lim_{x\to\infty} \ln\left((e^x+x)^{\frac{1}{x}}\right)} = e^1 = \boxed{e}$

#3) $y = xe^{-x^2}$

Intercepts: $y(0)=0$, so this is an x- and y-intercept.

When else is y zero? $0 = xe^{-x^2}$

$x=0$ or $e^{-x^2}=0 \leftarrow$ never happens

Asymptotes: $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \frac{\infty}{\infty}$ indeterminate

L'Hop $= \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = \frac{1}{\infty} = 0$.

$\lim_{x\to-\infty} xe^{-x^2} = \lim_{x\to-\infty} \frac{x}{e^{x^2}} = \frac{-\infty}{\infty}$ indeterminate

L'Hop $= \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} = \frac{1}{-\infty} = 0$

Two-sided HA at $y=0$.

The domain is all reals, so no VA.

CPs: $y' = 1e^{-x^2} + x\cdot(-2xe^{-x^2}) = (1-2x^2)e^{-x^2} = 0$

$1-2x^2=0$

$2x^2=1 \longrightarrow x^2=\frac{1}{2} \longrightarrow x=\pm\frac{1}{\sqrt{2}}$

		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
f'	$-$		$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

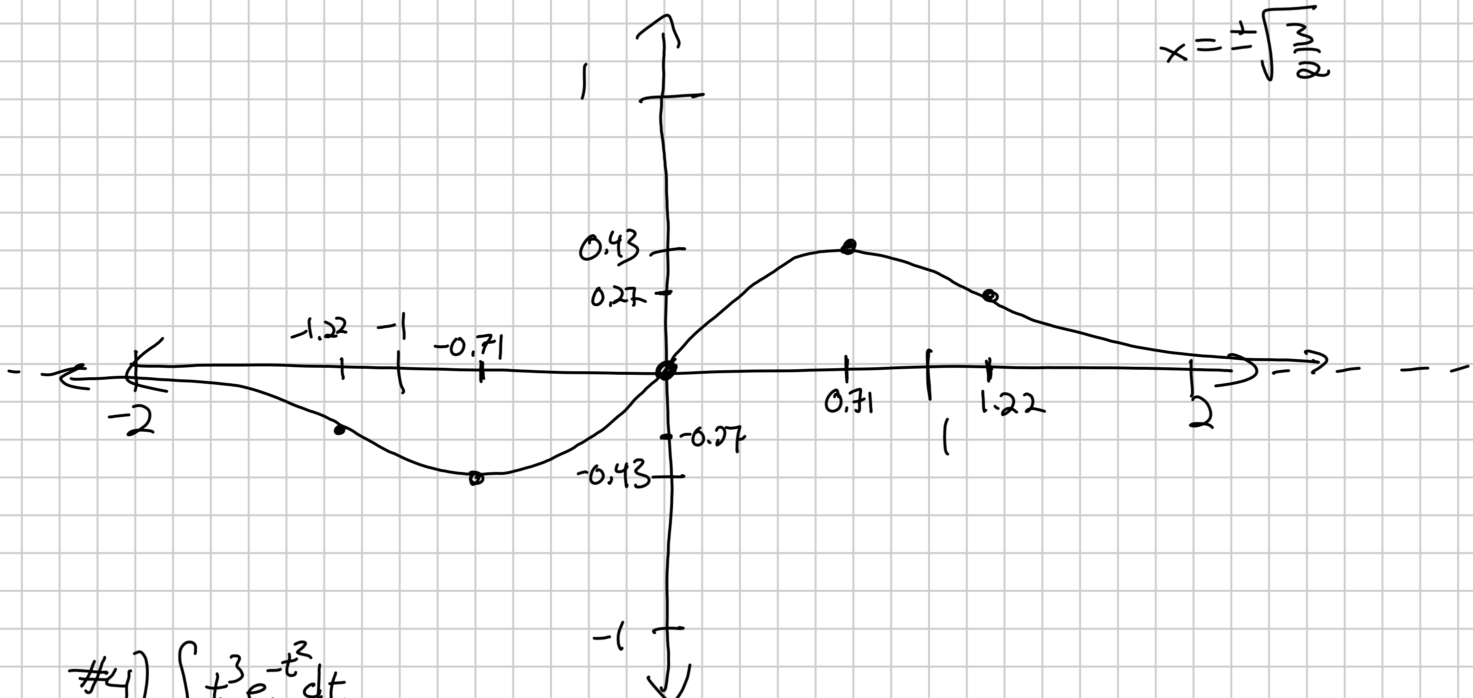
$f\left(\pm\frac{1}{\sqrt{2}}\right)=\pm\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}=\pm\frac{1}{\sqrt{2e}}$

<u>IPs</u>: $y''=-4xe^{-x^2}+(1-2x^2)(-2x)e^{-x^2}$

$=(-4x-2x+4x^3)e^{-x^2}$

$=2xe^{-x^2}(2x^2-3)$ $x=0$ or $2x^2=3$

$x=\pm\sqrt{\frac{3}{2}}$



#4) $\int t^3e^{-t^2}dt$

Let $u=-t^2$ $t^2=-u$

$=\int t\cdot t^2\cdot e^{-t^2}dt$ $du=-2t\,dt$

$=\frac{1}{2}\int(-t^2)e^{-t^2}(-2t\,dt)=\frac{1}{2}\int ue^u\,du$ IBP: $u=u$ $dv=e^u\,du$; $du=du$ $v=e^u$

$=\frac{1}{2}\left(ue^u-\int e^u\,du\right)=\frac{1}{2}(ue^u-e^u)+C=\frac{1}{2}\left(-t^2e^{-t^2}-e^{-t^2}\right)+C$

$=\boxed{\frac{-(1+t^2)e^{-t^2}}{2}+C}$

#5) $\displaystyle\int_{-\frac{\pi}{6}}^{\pi/6} e^{-\theta}\cos(2\theta)\,d\theta$

$u=\cos(2\theta) \qquad dv=e^{-\theta}d\theta$

$du=-2\sin(2\theta)d\theta \qquad v=-e^{-\theta}$

$$\int e^{-\theta}\cos(2\theta)d\theta = -\cos(2\theta)e^{-\theta} - \left(\int(-e^{-\theta})(-2\sin(2\theta))\,d\theta\right)$$

$$= -\left(\cos(2\theta)e^{-\theta} + 2\int e^{-\theta}\sin(2\theta)\,d\theta\right)$$

$u=\sin(2\theta) \qquad dv=e^{-\theta}d\theta$

$du=2\cos(2\theta)d\theta \qquad v=-e^{-\theta}$

$$= -\left(\cos(2\theta)e^{-\theta} + 2\left(-\sin(2\theta)e^{-\theta} - \int -e^{-\theta}(2\cos(2\theta)d\theta)\right)\right)$$

$$\int e^{-\theta}\cos(2\theta)d\theta = (-\cos(2\theta)+2\sin(2\theta))e^{-\theta} - 4\int e^{-\theta}\cos(2\theta)\,d\theta$$

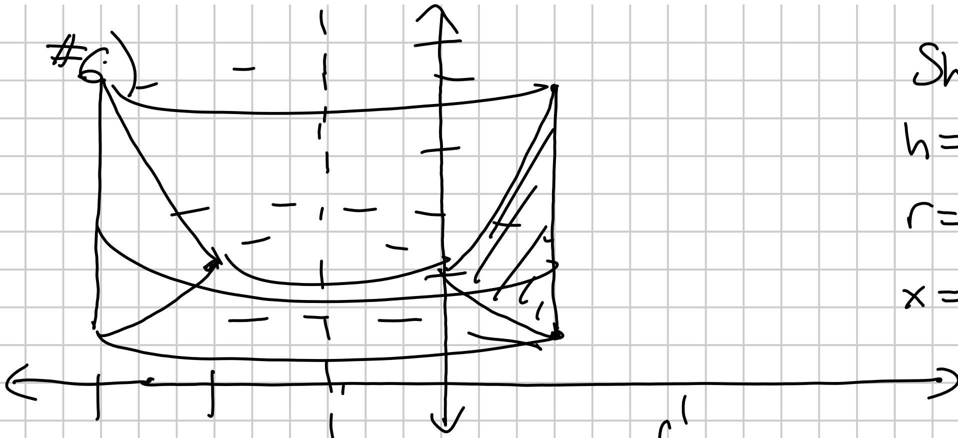
$$5\int e^{-\theta}\cos(2\theta)d\theta = (-\cos(2\theta)+2\sin(2\theta))e^{-\theta}$$

$$\int e^{-\theta}\cos(2\theta)d\theta = (-\cos(2\theta)+2\sin(2\theta))\frac{e^{-\theta}}{5}$$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{-\theta}\cos(2\theta)d\theta = \left(-\cos\left(\frac{\pi}{3}\right)+2\sin\left(\frac{\pi}{3}\right)\right)\frac{e^{-\pi/6}}{5} - \left(-\cos\left(\frac{\pi}{3}\right)-2\sin\left(\frac{\pi}{3}\right)\right)\cdot\frac{e^{\pi/6}}{5}$$

$$= \boxed{\left(\frac{2\sqrt{3}-1}{10}\right)e^{-\pi/6} + \left(\frac{1+2\sqrt{3}}{10}\right)e^{\pi/6}}$$

#6)



Shell method.

$h = e^{x} - e^{-x}$

$r = x+1$

$x = 0$ to 1

$$V = 2\pi \int_0^1 (1+x)(e^{x} - e^{-x})\,dx$$

$$= 2\pi \int_0^1 (e^{x} - e^{-x} + xe^{x} - xe^{-x})\,dx$$

$$\int xe^{x}\,dx = xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x} + C$$

$u = x \quad dv = e^{x}dx$

$du = dx \quad v = e^{x}$

$$-\int xe^{-x}\,dx = +xe^{-x} - \int e^{-x}\,dx = xe^{-x} + e^{-x} + C$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

So, $$V = 2\pi\left(\cancel{e^{x}} - \cancel{e^{-x}} + xe^{x} - \cancel{e^{x}} + xe^{-x} + \cancel{e^{-x}}\right)\Big|_0^1$$

$$= 2\pi\left(1\cdot e^{1} + 1\cdot e^{-1} - 0\cdot e^{0} - 0\cdot e^{-0}\right)$$

$$\boxed{V = 2\pi\left(e + \frac{1}{e}\right)}$$