Math 112
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Homework #6 Solutions

#1) a) $P(1900) = 76$ $\qquad P(t) = Ce^{at}$

$P(1910) = 92$ $\qquad 76 = Ce^{1900a}$

$92 = Ce^{1910a}$

Divide the second equation by the first,

$$\frac{92}{76} = e^{1910a-1900a} \qquad e^{10a}$$

$$10a = \ln\left(\frac{92}{76}\right) \qquad a = \frac{1}{10}\ln\left(\frac{92}{76}\right)$$

$$P(t) = Ce^{\frac{1}{10}\ln\left(\frac{92}{76}\right)t} = C\left(\frac{92}{76}\right)^{t/10}$$

$$P(1900) = C\left(\frac{92}{76}\right)^{\frac{1900}{10}} = 76$$

$$C = 76 \cdot \left(\frac{92}{76}\right)^{-190}$$

$$P(t) = 76\left(\frac{92}{76}\right)^{-190}\left(\frac{92}{76}\right)^{t/10}$$

$$P(t) = 76 \cdot \left(\frac{92}{76}\right)^{\frac{t-1900}{10}}$$

So, $P(2000) \cong 76 \cdot \left(\frac{92}{76}\right)^{\frac{2000-1900}{10}} \cong 76 \cdot \left(\frac{92}{76}\right)^{10}$

$= 513.5$ million

Obviously this is way larger than the actual population at 2000. This is probably because there was a big population surge during the 1900s, but in later decades the birth rate tapered off somewhat. Over time, that overestimate makes a huge difference in an exponential function.

b) $P(1990) = 250 = Ce^{1990a}$

$P(1980) = 227 = Ce^{1980a}$

As in (a), $e^{10a} = \frac{250}{227}$. $\qquad a = \frac{1}{10}\ln\left(\frac{250}{227}\right)$

$$P(t) = C \cdot \left(\frac{250}{227}\right)^{t/10}$$

$$227 = C \cdot \left(\frac{250}{227}\right)^{1980/10} \rightarrow C = 227 \cdot \left(\frac{250}{227}\right)^{-1980/10}$$

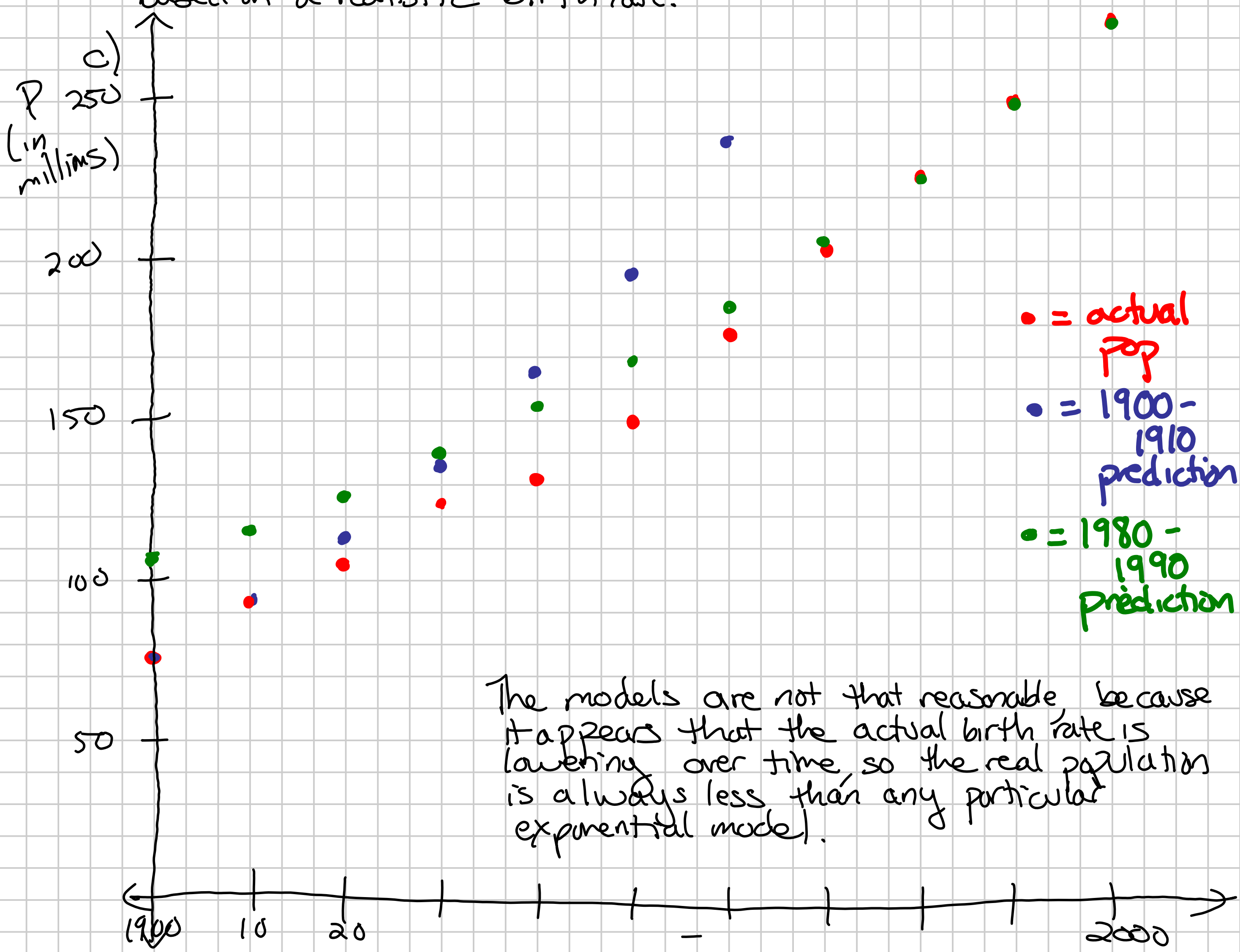
So, $P(t) = 227 \cdot \left(\frac{250}{227}\right)^{-1980/10} \left(\frac{250}{227}\right)^{t/10}$

$$= 227 \left(\frac{250}{227}\right)^{\frac{t-1980}{10}}$$

$$P(2000) \sim 227 \left(\frac{250}{227}\right)^{\frac{2000-1980}{10}} = \frac{250^2}{227} = 275.3 \text{ million}$$

This is actually pretty accurate which is not surprising since it's based on recent census data and thus is likely to be based on a realistic birth rate.

c)



The models are not that reasonable, because it appears that the actual birth rate is lowering over time, so the real population is always less than any particular exponential model.

#2) Let $M(t)=$ the amount owed at time t.

$$\frac{dM}{dt} = .005M - k$$

because the monthly interest rate is 0.5%. Here k represents the monthly payment, which we are solving for.

Let $y(t) = M - \frac{k}{0.005}$

$$y + \frac{k}{.005} = M$$

$$\frac{dy}{dt} = \frac{dM}{dt} = .005M - k$$
$$= .005\left(y + \frac{k}{.005}\right) - k$$
$$= .005y + \cancel{k - k}$$

So, $\frac{dy}{dt} = .005y$

$$y = Ce^{.005t}$$
$$M(t) = Ce^{.005t} + \frac{k}{.005}$$

$M(0) = 15,000$ and $M(60) = 0$ to pay it off in 5 years $= 60$ months.

$$M(0) = C + \frac{k}{.005} = 15000$$

$$\frac{1}{.005} = \frac{1}{\frac{5}{1000}} = 200$$

$$C = 15000 - \frac{k}{.005}$$

$$M(t) = (15000 - 200k)e^{t/200} + 200k$$

$$M(60) = 0$$

$$0 = (15000 - 200k)e^{60/200} + 200k$$

$$15000e^{3/10} = 200ke^{3/10} - 200k$$
$$= k \cdot 200(e^{3/10} - 1)$$

$$k = \frac{150\cancel{00}}{2\cancel{00}(e^{3/10} - 1)} = \boxed{\$214.37}$$

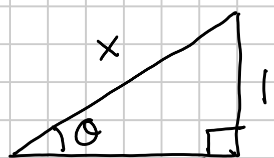
If the interest were compounded monthly instead of continuously, the total interest would compound more slowly, i.e. the effective interest rate would be lower (but only a little). This would slightly lower the necessary monthly payment.

3) Find $(\csc^{-1})'(x)$: $\csc'(x) = -\csc(x)\cot(x)$

Therefore $(\csc^{-1})'(x) = \dfrac{-1}{\csc(\csc^{-1}(x))\cot(\csc^{-1}(x))}$

We know that $\csc(\csc^{-1}(x)) = x$.

What about $\cot(\csc^{-1}(x))$?



$\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{H}{O}$

By Pythagoras, $\sqrt{x^2-1}$

So, in our triangle, $\csc(\theta) = x$

and $\theta = \csc^{-1}(x)$

Then $\cot(\csc^{-1}(x)) = \cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{A}{O}$

$= \dfrac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$

So, $(\csc^{-1})'(x) = \dfrac{1}{\csc(\csc^{-1}(x)) \cdot \cot(\csc^{-1}(x))}$

$= \boxed{\dfrac{-1}{x\sqrt{x^2-1}}}$

4) $f(x) = \sin^{-1}(e^x)$.

Domain: The domain of $\sin^{-1}(u)$ is $[-1,1]$

When is $-1 \le e^x \le 1$? e^x is always ≥ 0

$e^x \le 1 \Rightarrow x \le \ln 1 = 0$

The domain of e^x is unrestricted.
So, the domain of the whole function is $(-\infty, 0]$

Range: When $x \to -\infty$, $e^x \to 0$ and $\sin^{-1}(0) = 0$, so the lower end of the range is 0 (not included).
On the other hand, $\sin^{-1}(e^0) = \sin^{-1}(1) = \frac{\pi}{2}$.

So, the range is $(0, \frac{\pi}{2}]$.

Derivative: $f'(x) = \dfrac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \boxed{\dfrac{e^x}{\sqrt{1-e^{2x}}}}$

Graph: Note that for $x \le 0$, $e^x \le 1$, f' is always ≥ 0, so f is always increasing. But $f'(x) \to \infty$ as $x \to 0$.

Also, $f''(x) = \dfrac{-e^x \frac{1}{2\sqrt{1-e^{2x}}} \cdot (-2e^{2x}) + \sqrt{1-e^{2x}} \cdot e^x}{(1-e^{2x})}$

$$= \frac{2e^{3x} + e^x - e^{3x}}{(1-e^{2x})^{3/2}} = \frac{e^{3x} + e^x}{(1-e^{2x})^{3/2}} > 0,$$

so f is always concave up.