Math 112
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Homework #4 Solutions

#1) a) The hare's distance traveled as a function of time is

$$H(T) = \int_0^T v(t)\,dt = \int_0^T \frac{5\pi}{4}\sin\left(\frac{\pi t}{400}\right)dt \qquad u = \frac{\pi t}{400}$$
$$du = \frac{\pi}{400}\,dt$$

$$\int \frac{5\cancel{\pi}}{\cancel{4}} \cdot \frac{\cancel{400}^{\,1}}{\cancel{\pi}} \sin(u)\,du = -500\cos(u) + C = -500\cos\left(\frac{\pi t}{400}\right) + C$$

$$H(T) = -500\cos\left(\frac{\pi t}{400}\right)\Big|_0^T = -500\cos\left(\frac{\pi T}{400}\right) + 500\cos(0)$$
$$= 500 - 500\cos\left(\frac{\pi T}{400}\right)$$

When does the hare cross the finish line?

$$1000 = 500 - 500\cos\left(\frac{\pi T}{400}\right)$$
$$500 = -500\cos\left(\frac{\pi T}{400}\right)$$
$$\cos\left(\frac{\pi T}{400}\right) = -1 \qquad \frac{\pi T}{400} = \pi \quad \to T = 400 \text{ seconds}$$

The turtle's distance traveled as a function of time is

$$D(T) = \int_0^T 3\,dt = 3t\Big|_0^T = 3T.$$

When does he cross the finish line? $3T = 1000$

$$T = \frac{1000}{3} \approx 333.3\,s$$

The turtle wins! (because he gets to the finish line first)

b) The hare's maximum speed occurs when $v'(t) = 0$

$$v'(t) = \frac{5\pi^2}{1600}\cos\left(\frac{\pi t}{400}\right) = 0 \qquad \frac{\pi t}{400} = \frac{\pi}{2} \to t = 200$$

$$v(200) = \frac{5\pi}{4}\sin\left(\frac{\pi}{2}\right) = \frac{5\pi}{4} \approx \boxed{3.927\ m/s}$$

The hare's maximum speed is greater than the turtle's speed, so if it had run at that speed steadily then, yes it would have won.

The race ends at $T = \frac{1000}{3}$, so the hare's average speed during the race is:

$$V_{ave} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{3}{1000}\int_0^{\frac{1000}{3}} \frac{5\pi}{4}\sin\left(\frac{\pi t}{400}\right)dt$$

$$= \frac{-15\cancel{\pi}}{\cancel{4000}_{10}} \cdot \frac{\cancel{400}}{\cancel{\pi}}\cos\left(\frac{\pi t}{400}\right)\Big|_0^{\frac{1000}{3}} = -\frac{3}{2}\cos\left(\frac{1000\pi}{1200}\right) - -\frac{3}{2}\cos(0)$$

$$= \frac{3}{2}\left(1 - \cos\left(\frac{5\pi}{6}\right)\right) = \frac{3}{2}\left(1 + \frac{1}{2}\right) = \frac{9}{4} = \boxed{2.25\ m/s}$$

c) The hare achieves its average speed if

$$\frac{9}{\cancel{4}} = \frac{5\pi}{\cancel{4}}\sin\left(\frac{\pi t}{400}\right)$$

$$\sin\left(\frac{\pi t}{400}\right) = \frac{9}{5\pi} \rightarrow \frac{\pi t}{400} = 0.61011 \text{ or } 2.53148$$

So, $\boxed{t = 77.682\,s \text{ and } 322.318\,s}$

2) $f(x) = 2 + x^3 + \sin(x)$

$f'(x) = 3x^2 + \cos(x)$ Can this ever be 0?

For $\frac{-1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}$, $\cos(x) > 0$

and for $x > \frac{1}{\sqrt{3}}$ or $x < \frac{-1}{\sqrt{3}}$, $3x^2 > 1$ and $|\cos x| \le 1$

So the total is always strictly positive and $\boxed{\text{yes}}$, f is one-to-one.

3) $f(x) = \sqrt{x^3+8}$. a) Domain? We need what's under the square root to be nonnegative: $x^3 + 8 \ge 0$

$$x^3 \ge -8$$
$$x \ge -2.$$

Range? The output of the square root is all nonnegative numbers.

For the inverse function, the domain and range get switched, so:

$\boxed{\text{The domain of } f^{-1} \text{ is all nonnegative numbers.}}$

$\boxed{\text{The range of } f^{-1} \text{ is all numbers greater than or equal to } -2.}$

b) $y=f^{-1}(4)$ if $\quad 4=f(y)$

$$4=\sqrt{y^3+8} \qquad 16=y^3+8$$

$$8=y^3 \rightarrow \boxed{y=2}$$

c) $(f^{-1})'(3)$. First, we need to know what $f^{-1}(3)$ is:

$$3=\sqrt{y^3+8} \qquad 9=y^3+8$$

$$1=y^3 \rightarrow y=1$$

Also, $f'(x)=\dfrac{1}{2\sqrt{x^3+8}}\cdot 3x^2=\dfrac{3x^2}{2\sqrt{x^3+8}}$

So, $f'(1)=\dfrac{3\cdot 1^2}{2\sqrt{1^3+8}}=\dfrac{3}{2\cdot 3}=\dfrac{1}{2}$

Thus, $(f^{-1})'(3)=\dfrac{1}{f'(f^{-1}(3))}=\dfrac{1}{f'(1)}=\dfrac{1}{\frac{1}{2}}=\boxed{2}$

#4) Explain what an inverse function is. How can you tell whether a given function has an inverse or not? How can you find the values of the inverse function?

An inverse function is a function that reverses the action of a given function. In other words, if you start with a function y=f(x), the inverse function $f^{-1}(x)$ is the unique function such that, for all x in the domain of f, $f^{-1}(f(x))=x$, and, for all y in the range of f, $f(f^{-1}(y))=y$. A given function f(x) only has an inverse function if it is one-to-one. In other words, f must hit each value in its range only once. One can tell if a function is one-to-one in three ways: if one plots the function and it passes the horizontal line test (Each horizontal line crosses the graph at most once.); if one can check algebraically that two different x values always produce different y values; or if the function is differentiable, and its derivative is either always strictly negative or always strictly positive. If a function is one-to-one, then it always has an inverse on some subset of the reals. One can find the inverse in two possible ways. If one has a graph of the function, one can get a graph of the inverse function by reflecting the original graph across the line y=x. If one has a formula for the function y=f(x), one finds a formula for the inverse function by switching y and x and then solving for y algebraically.