Math 112
Dr. Raynor
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Homework #2 Solutions

#1) $$g(x) = \int_{\sec(x)}^{x^3} \frac{1}{\sqrt{1+t^2}}\,dt = \int_{\sec(x)}^{0} \frac{dt}{\sqrt{1+t^2}} + \int_0^{x^3} \frac{dt}{\sqrt{1+t^2}}$$

$$= -\int_0^{\sec(x)} \frac{dt}{\sqrt{1+t^2}} + \int_0^{x^3} \frac{dt}{\sqrt{1+t^2}}$$

Find $\frac{d}{dx}\int_0^{\sec(x)} \frac{dt}{\sqrt{1+t^2}}$. Let $u = \sec(x)$

$$\frac{du}{dx} = \sec(x)\tan(x)$$

$\int_0^{\sec(x)} \frac{dt}{\sqrt{1+t^2}} = \int_0^{u} \frac{dt}{\sqrt{1+t^2}}$, so, by the FTC, $\frac{d}{du}\int_0^{u} \frac{dt}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+u^2}}$

Hence, by the chain rule, $$\frac{d}{dx}\int_0^{\sec(x)} \frac{dt}{\sqrt{1+t^2}} = \frac{d}{du}\left(\int_0^{u} \frac{dt}{\sqrt{1+t^2}}\right)\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1+u^2}} \cdot \sec(x)\tan(x)$$

$$= \frac{\sec(x)\tan(x)}{\sqrt{1+\sec^2 x}}$$

Similarly, $\frac{d}{dx}x^3 = 3x^2$, so by the FTC and chain rule,

$$\frac{d}{dx}\int_0^{x^3} \frac{dt}{\sqrt{1+t^2}} = \frac{d}{du}\left(\int_0^{u} \frac{dt}{\sqrt{1+t^2}}\right)\frac{du}{dx} = \frac{1}{\sqrt{1+u^2}} \cdot 3x^2 = \frac{3x^2}{\sqrt{1+(x^3)^2}}$$

Plugging Back In, $$\frac{dg}{dx} = \boxed{\frac{3x^2}{\sqrt{1+x^6}} - \frac{\sec(x)\tan(x)}{\sqrt{1+\sec^2 x}}}$$

#2) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{2i}{n}}$ represents a Riemann sum.

$\Delta x = \frac{2}{n}$, so $b-a=2$

$f(x_i) = \sqrt{1+\frac{2i}{n}}$, so $f(x) = \sqrt{x}$ and $x_i = 1 + \frac{2i}{n}$,

so $a=1$ and $b=3$.

Hence, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{2i}{n}} = \int_1^3 \sqrt{x}\,dx$

We know that the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x^{3/2}+C$,
so by the FTC, $\int_1^3 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^3 = \boxed{\frac{2}{3}3^{3/2} - \frac{2}{3}}$

#3) $a(t) = \sin(t)\, m/s^2$ $\quad t=0$ to 2π $\quad v(0) = -\frac{1}{2}$

a) $v'(t) = a(t)$, so $v(t) = \int a(t)\,dt + C$

$$= \int \sin(t)\,dt + C$$

$$v(t) = -\cos(t) + C$$

$$v(0) = -1 + C = -\frac{1}{2} \Rightarrow C = \frac{1}{2}$$

So, $v(t) = \boxed{-\cos(t) + 3/2\, m/s}$

b) The net change in position is the definite integral of velocity:

$$\Delta x = \int_0^{2\pi} \left(-\cos(t) + \frac{1}{2}\right) dt = \left(-\sin(t) + \frac{1}{2}t\right)\Big|_0^{2\pi}$$

$$= -0 + \pi + 0 - 0 = \boxed{\pi\, m}$$

c) The total distance traveled is the integral of the absolute value of velocity.

$|v(t)| = \left|-\cos(t) + \frac{1}{2}\right|$. When is v negative?

$$-\cos(t) + \frac{1}{2} = 0 \rightarrow \cos(t) = \frac{1}{2} \quad t = \frac{\pi}{3}, \frac{5\pi}{3}.$$

So, Distance $= \int_0^{\pi/3} \left(\cos(t) - \frac{1}{2}\right) dt + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(-\cos(t) + \frac{1}{2}\right) dt$

$$+ \int_{\frac{5\pi}{3}}^{2\pi} \left(\cos(t) - \frac{1}{2}\right) dt$$

$$= \left(\sin(t) - \frac{t}{2}\right)\Big|_0^{\pi/3} + \left(-\sin(t) + \frac{t}{2}\right)\Big|_{\pi/3}^{5\pi/3} + \left(\sin(t) - \frac{t}{2}\right)\Big|_{\frac{5\pi}{3}}^{2\pi}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} - 0 + 0 + \frac{\sqrt{3}}{2} + \frac{5\pi}{6} - -\frac{\sqrt{3}}{2} - \frac{\pi}{6} + 0 - \pi - -\frac{\sqrt{3}}{2} + \frac{5\pi}{6}$$

$$= \boxed{\left(2\sqrt{3} + \frac{4\pi}{3}\right) m}$$

#4) $\displaystyle\int_0^1 x^3\sqrt{x^2+1}\,dx$ $\qquad u = x^2+1$, $du = 2x\,dx$ $\qquad x^2 = u-1$, $x=0 \to u=1$, $x=1 \to u=2$

$$= \frac{1}{2}\int_0^1 x^2\sqrt{x^2+1}\,(2x\,dx) = \frac{1}{2}\int_1^2 (u-1)\sqrt{u}\,du$$

$$= \frac{1}{2}\int_1^2 \left(u^{3/2} - u^{1/2}\right)du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^2$$

$$= \frac{1}{2}\left(\frac{2}{5}\cdot 2^{5/2} - \frac{2}{3}\cdot 2^{3/2} - \frac{2}{5} + \frac{2}{3}\right)$$

$$= \frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3} = \frac{12\sqrt{2} - 10\sqrt{2} - 3 + 5}{15} = \boxed{\frac{2}{15}\left(1 + \sqrt{2}\right)}$$

#5) When we say that the derivative and the integral are inverse processes, what we mean is that each one reverses the action of the other. This is true because of the fundamental theorem of calculus. The FTC part 1 says that the derivative of the definite integral of a function, which the top limit of integration as the variable, gives you the original function back. In other words, the derivative reverses the action of the definite integral. On the other hand, the FTC part 2 says that if you take the definite integral of the derivative of a function, you will get the original function back, evaluated at the endpoints. In other words, the definite integral reverses the action of the derivative, almost. It’s not an exact reverse because you have to evaluate at both endpoints. Thus, the fundamental theorem of calculus states that the derivative reverses the integral and the integral almost reverses the derivative, and that’s why the derivative and the integral are considered inverse processes.

#6) a) FALSE. All differentiable functions are continuous, but the opposite is not always true. For example, the function $f(x) = \sqrt[3]{x}$ is continuous for all reals, but it does not have a derivative at $x=0$.

b) TRUE. The fundamental theorem of calculus part I says that for any continuous function $f(x)$, the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, as long as f is continuous at a, so f has at least one antiderivative.