Math 331 Project List
due Sat., Dec. 12th, 9am

Report on a geometric topic of your choosing. You should complete a 6-10 page, typed mathematical paper. Your paper should have a substantial mathematics component, which could include proofs, definitions, and sketches. Proofs may be adapted from other sources or may be your own original work. Students in 631 should complete a more substantial project than 331 students.

In addition, prepare a 15-20 minute presentation on your work. Currently, the presentations are scheduled 9-12 on Saturday, Dec. 12th. Your presentation may utilize overhead slides, poster, or an electronic format.

Below, over 30 possible topics are listed. By November 1st, you should choose one of them to explore, and let me know which one you are investigating. Or you may propose your own topic (subject to my approval). For most of these topics, I can suggest some outside resources for you to pursue. Some are given below; see the back of Greenberg's text for full bibliographic information.

Project Topics

1. Consider one or more of the attempts to prove the Euclidean parallel postulate. Report on the history, on the flaws, and historical significance of the attempt(s). We will summarize several of these attempts from chapter 5.

2. Report on other proofs of the Pythagorean Theorem (in Euclidean geometry), besides those given in the book. You may want to consider the proof attributed to President Garfield. (See Maor, 2007).

3. (a) Investigate hyperbolic geometry's influence upon artists, such as M.C. Escher.
   (b) Consider tessellations (repeating patterns) occurring in the hyperbolic plane; you should go in much more depth than we will in §6.2.
   (You can do one of these or combine them. See the p.197 for an example. Students do not seem to do well with this project, so make sure yours is a *math* project if you choose this one.)

4. Jeff Weeks, a MacArthur 'genius grant' recipient who visited Wake Forest in fall 2007, discussed different geometric structures which could occur in the shape of space. Examine this topic further. I can provide lots of references, including Weeks's book, *Shape of Space*.

5. Victorian and late 19th-century England strongly resisted the notions of non-Euclidean geometry. See Joan Richards, *Mathematical Visions: The Pursuit of Geometry in Victorian England*, 1988.

6. Helmholtz and Clifford, wonderfully talented 19th century mathematicians, helped popularize non-Euclidean geometry. Clifford's work was a precursor to Einstein's general relativity. Report on their contributions.

7. in Euclidean geometry Gauss showed the following: a regular n-gon is constructible if and only if the prime factorization of n consists of 2 to some power times a product of odd distinct *Fermat primes*, all to the first power. A Fermat prime is one of the form

$$F_m = 2^{2^m} + 1, \qquad m \geq 0$$

Currently only 5 Fermat numbers are known to be prime, meaning only 31 odd values of n produce constructible n-gons. Investigate and describe this result.

8. (Archimedes Axiom and the Sacchieri-Legendre theorem)

9. Investigate one of the following mathematician's contributions to non-Euclidean geometry in a detailed, mathematically rigorous treatment:
(a) Ptolemy (b) Proclus (c) Nasir Eddin al-Tusi (d) John Wallis (e) Girolamo Sacchieri
(f) Alexis Clairaut (g) Legendre (h) Lambert (i) F.A. Taurinus (j) Hilbert

10. Constructing perpendicular lines with a ruler only, via Hilbert's ideas

11. Feuerbach circle: given a triangle, there is a special circle associated to it, which contains 9 important geometric points: the midpoints of each side, the base of the altitudes and other points on the altitudes. Investigate the history and importance of this circle.

12. Desargues' Theorem (Greenberg ch. 2, project 1)

13. Investigate plane projective geometry further (Greenberg, ch. 2, project 2)

14. Explore the geometric transformations in detail for one of our primary three models (K, H, D) of hyperbolic space. There is a group structure associated to maps which take one model bijectively to itself. Lots of pictures should occur in this project.

15. Beltrami-Klein coordinates (see Greenberg ch. 10)

16. Morley's Theorem: The angle trisectors of a triangle intersect to form an equilateral triangle. Discuss its proof. Is it true in hyperbolic geometry?

17. Trigonometry. Spherical trig and hyperbolic trig. What is the analog of the Pythagorean Theorem?

18. Consider the sphere S which forms a model of hyperbolic geometry: points are the southern hemisphere points, and lines are Euclidean semicircles, centered on the equator. Explore all of the different stereographic projections from S onto a plane: by varying the point p where the plane is tangent to S, we get different hyperbolic models. (e.g., Poincare disk, upper-half plane)

19. Explore 3-dimensional hyperbolic geometry; contrast with 3-d Euclidean space.

20. Right polygons in hyperbolic space. Can you make them regular?

21. Circles of Apollonius

22. Produce excellent computer graphics of various hyperbolic models. Utilize them to illustrate selected hyperbolic theorems and concepts.

23. Compile a long list of statements equivalent to Euclid's parallel postulate. Pick some interesting and/or nonintuitive ones and prove their equivalence.

24. Finding the symmetry point S in hyperbolic space between lines ℓ, n. (Greenberg, ch.6, project 1)

25. Develop calculus on a hyperbolic geometry. Develop the formulas for arclength and area, using polar coordinates and Lobachevsky coordinates. (see Greenberg ch. 10)

26. Begin with the right triangle construction theorem from ch. 10. Then show, given any three angle measures, α, β, γ which sum to less than $180°$, that we can construct a triangle in the hyperbolic plane having these 3 angle measures. (See Martin, 1982, p. 486.)

27. Take a singly asymptotic triangle, i.e., one vertex is an ideal point. Show that its altitudes are concurrent, and that its angle bisectors are concurrent. Explore singly asymptotic triangle further, or doubly asymptotic triangles.

28. Discuss the geometry of ideal triangles: all vertices are ideal points (which we study in hyperbolic geometry).

29. Cayley-Klein geometries. These are founded by determining angles and segments relative to a distinguished conic – the absolute, which can be real or imaginary. If real, we get metric formulas which produce hyperbolic geometry. If imaginary, we get elliptic geometry. (see Yaglom, *A Simple non-Euclidean Geometry and Its Physical Basis*, 1979.)

    Cayley-Klein geometries are important in quantum physics and also in engineering. You might also report on these connections.

30. Report on Hilbert's arithmetic of *ends*. Hilbert consturcts a field that can be used to put coordinates on a general hyperbolic plane, and then perform analytic geometry and calculus upon it.

31. Investigate the intertwining of hyperbolic geometry with special relativity.

32. (for students who have a taken an algebra course) Define a *Euclidean plane* to be one that obeys Hilbert's parallel axiom, the 13 axioms of incidence, betweenness, and congruence, and the circle continuity principle (rather than Dedekind's Axiom). What are the possible Euclidean planes? They turn out to be F^2, where F is a Euclidean field. See projects 2 & 3 in chapter 3.

    If we drop the circular continuity principle, the possible planes are again described by F^2, where F is a Pythagorean field. Equilateral triangles still exist. Explore why. We can't use our proof of Euclid's 1st proposition – why? But, since $\sqrt{3} \in F$, there is an algebraic proof.

    One (somewhat difficult) reference is Hartshorne, *Geometry: Euclid & Beyond* (2000), section 21. I can help you find others.

33. Fuchsian groups and Kleinian groups – what are they? How do they relate to hyperbolic geometry?

34. Write an essay on the development of geometry in ancient Greece. You may want to explore Pythagoras and his followers, who believed 'all is number', and/or the female mathematician Hypatia.

35. Though both Poincare and Klein have models of hyperbolic geometry named for them, they were not friends. Investigate, mathematically, their rivalry.

36. Starting with the book by O'Shea, report on the Poincare Conjecture. Focus on its history and Poincare's ideas (not on the recent proof, which is way too difficult for our scope).

37. Any suitable neat theorem in Euclidean or non-Euclidean geometry.