Chapter 2: Knot Theory

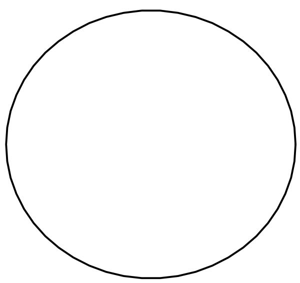
Wednesday, February 15, 2012
10:06 AM

Section 2.1: Knots, Links and Equivalences

The official defn. (in our book) of a knot is somewhat surprising: we define it as a polygon.

Defn a **knot** K is a simple closed curve in $\mathbb{R}^3$ that can be broken up into a finite number of edges $e_1, e_2, \cdots, e_n$. Note that e_i and e_{i+1} meet only at the appropriate endpoint, as do e_1 and e_n; otherwise the edges do not intersect.

"But knots don't have to be polygons!", you might object. And that's fair. We could define a knot as "a simple closed curve in $\mathbb{R}^3$". But our definition is (almost) equivalent to this.

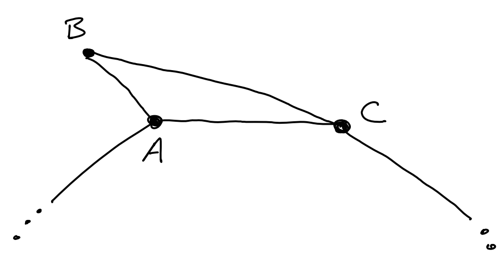


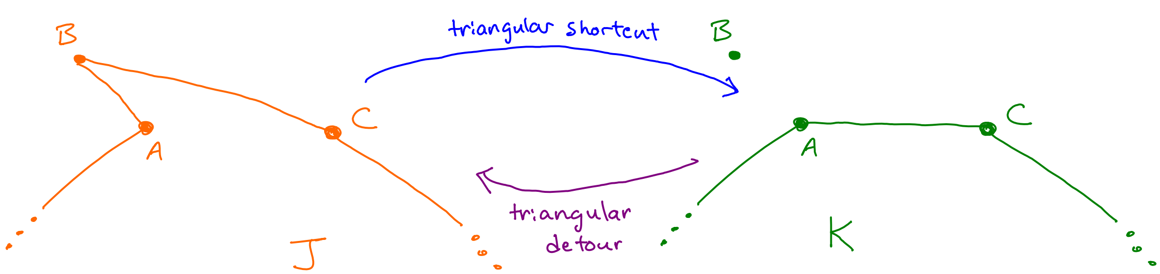
Approximate via an octagon
a dodecagon
a 100-gon
a 1000-gon
a 1,000,000-gon
⋮

We will draw knots as smooth curves, still.

So, how do we tell knots apart?

Defn We can take 1 edge of a knot and make it 2 edges by adding a point B. (e.g., go from K to J)
Or take 2 edges + "cut the corner" and make it 1
(e.g. going from J to K)





Wednesday, February 15, 2012
10:27 AM

We call the former, going from K to J, a triangular detour.

We call the latter, going from J to K, a triangular shortcut.

We say two knots are equivalent if there is a (finite) sequence of triangular moves.

Proposition This is an equivalence relation on knots.

Proof: Reflexive $K \sim K$ no moves needed

Symmetry If $J \sim K$, then take sequence of moves, start at K, and undo them to get to J. Every triangular detour from J to K needs a triangular shortcut going from K to J, and vice-versa.

Transitive "add" sequences: first go J to K then K to L to get from J to L.

Defn the equivalence classes of knots are known as knot types (or knotclasses).

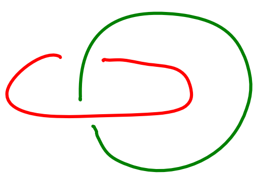
All triangular moves are ambient isotopies.

All ambient isotopies can be approximated (as close as you like) by a sequence of triangular moves.

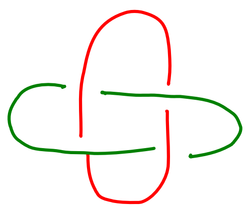
$\therefore$ We may use either triangular moves or ambient isotopies, as we like.

Defn a link is a disjoint collection of knots

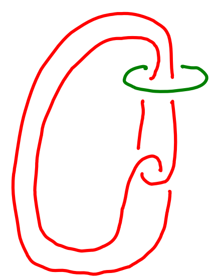
Examples

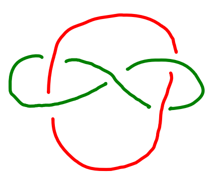


Hopf link 2^2_1



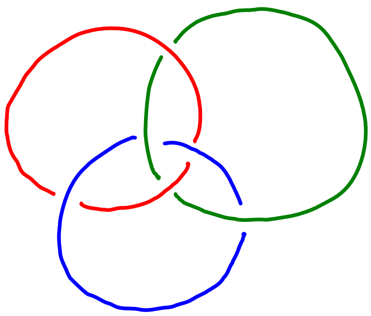
Solomon's seal 4^2_1





two diagrams of the Whitehead link 5^2_1

Friday, February 17, 2012
10:19 AM

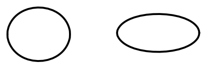


Borromean rings 6^3_2

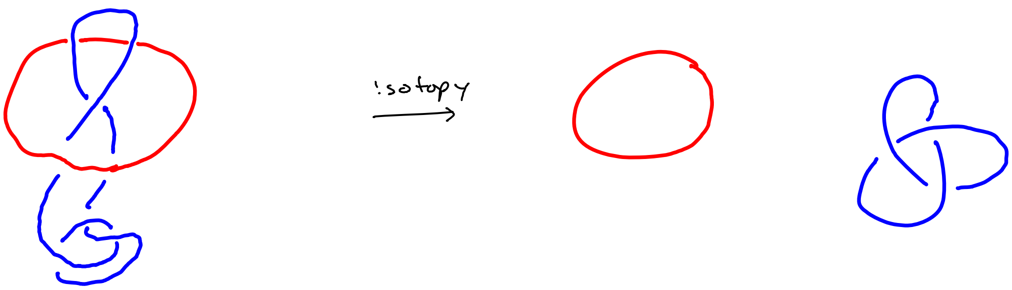
The Borromean rings have this property — they cannot be pulled apart, but if any one component is removed, they will split apart.

Defn an unknot is any knot equivalent to a circle

an unlink is any link equivalent to



a split link separates into 2 pieces, i.e., we may draw a sphere (which separates $\mathbb{R}^3$) with one piece inside and one outside



§2.2 Knot Diagrams

general position

$$\dim(A \cap B) = \dim A + \dim B - n$$

codimension

$$\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$$

book – polyhedra

Knot diagrams

project

triple point

Q: ~~How~~ How to describe possible directions of a projection? S^2

trisecant

quadrisecant

Defn The orthogonal projection of a knot onto a plane is a regular projection

if (1) no vertex projects to a double point

(2) $\nexists$ triple points

Knot diagram – resolve crossings

Defn knot invariant is a mathematical property associated to a knot that does not change under ambient isotopy / triangular moves of knot.

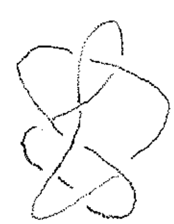
Ex: minimal crossing number of K

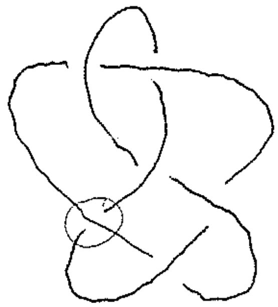
$$Cr(K) = \min \# \text{ of crossings among all diagrams of } K$$

~~Different~~ The same projection can lead to different knot diagrams

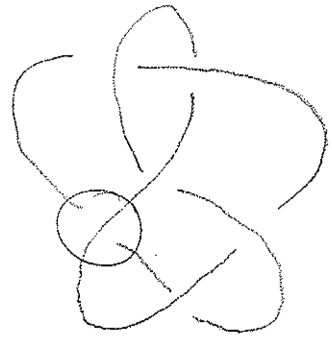
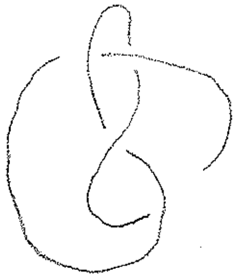
3_1

0_1





6_2

 $\simeq$ 

4_1

If a projection has n crossings, explain why there are ~~at most~~ 2^n different knot diagrams that result.

Ex: Show there is only one knot type with $Cr(K)=0$.

Consider K with $Cr(K)=0$. So it has a projection with no crossings

– this forms a s.c.c. in the plane.

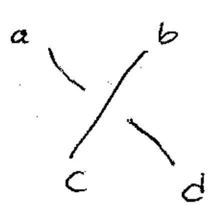
By Exercise 2.2.10, we may use triangular moves to get all vertices in the plane.

Now, round out this polygon to a round circle, the unknot .

(ambient isotopy approach)

Or, use 1.6.11
1.6.12 $\rightsquigarrow$ triangular moves make this a triangle.

Ex. $\nexists$ knot with $Cr(K)=1$.



Pairs

a-b
c-d  $\simeq$ 

a c
b d  $\simeq$ 

a d
b c  creates new crossing

Defn alternating knot

1877 Peter Guthrie Tait showed all knots with $Cr(K) \leq 7$ (15 of them) were alt.

- found some nonalt. examples

1930 Bankwitz proved these were nonalt.

Defn reduced diagram # crossings cannot be reduced by...

more precisely, draw a topological circle around K_1. Does flipping it over reduce #crossings?

Defn reduced alternating diagram

Theorem (Kauffman - Murasugi - Thistlethwaite)

If you have a reduced alt. diagram of K, then $Cr(K) =$ its # crossings.

YOUR PET KNOT

We will each pick an 8-crossing knot. As we progress, you will be asked ~~[illegible]~~ to record the value of all knot invariants we find for your knot.

Many hw questions.

At conclusion - hand in a document.