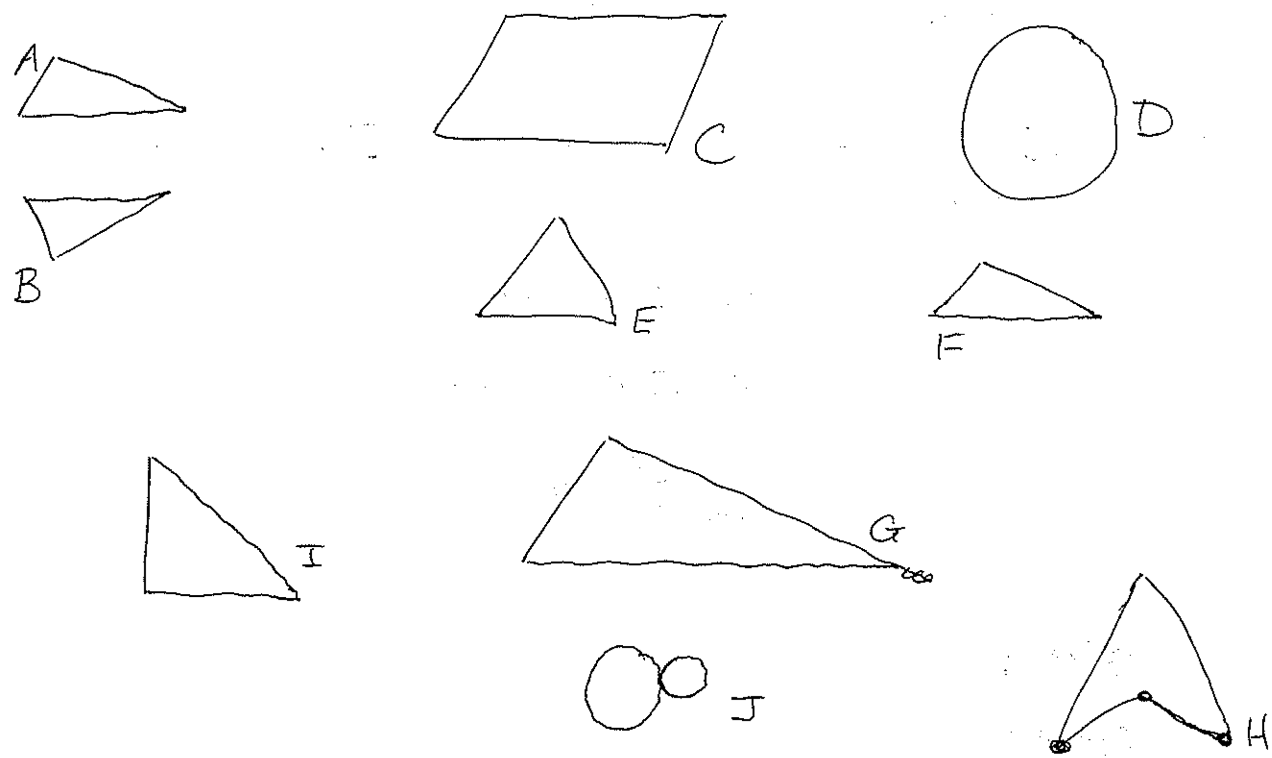
CHAPTER 1 : DEFORMATIONS

§1.1 Equivalence Relations

which of these objects are the same as A?



It depends on what we mean by the same?

- congruent   A, F, B
- similar   A, B, F, G
- three-sided   A, B, F, G, I
- convex   A – G, I
- polygons   A, B, C, E – I

a binary relation $\square$ relates two objects ~~between~~ in the same set.

Defn ~~equivalence relation~~

For example, $<$ is a binary relation on the real numbers.

$2 < 3$ $\quad$ $3 \not< 2$

$-5 < 0$ $\quad$ $2 \not< 2$

Ex: $\square$ on people where $a \square b$ if a is a descendant of b

me $\square$ my dad

me $\square$ my grandmother

my dad $\not\square$ me

reflexive

symmetric

transitive

equivalence relation

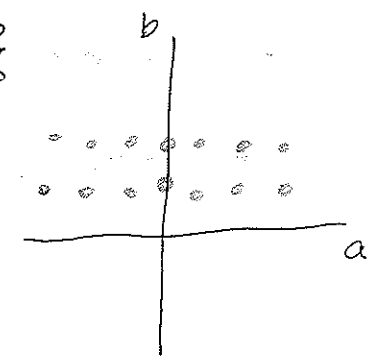
Ex: $\mathbb{Z}$ mod 10

shirt color

consider set $X = \{(a,b) : a,b \in \mathbb{Z},\ b \neq 0\}$

$(a,b) \sim (c,d)$ if $ad = bc$

equality



HW: Find 3 examples of equivalence relns

- at least 1 mathematical
- at least 1 not.

Math 361. Chapter 1: Deformations
section 1.1: Equivalence Relations

Friday, January 20, 2012
9:46 AM

(see pages 1-2, handwritten)

Ex: Consider the set $\mathbb{Z} \times \mathbb{Z} = \{ (a,b) : a,b \in \mathbb{Z} \}$
and relation $(a,b) \sim (c,d)$ if $ad = bc$.

Is this an equivalence relation?

Reflexivity + Symmetry still hold.
However, transitive property doesn't:
$(1,1) \sim (0,0)$
$(0,0) \sim (1,0)$
but $(1,1) \not\sim (1,0)$.

HW ① Consider the following argument:

claim If relation $\sim$ on a set X is both symmetric and transitive, then it is reflexive.

why? $x \sim y$ implies $y \sim x$ by symmetry. Now apply the transitive property:
since $x \sim y$ and $y \sim x$, we conclude $x \sim x$.

This claim is **FALSE**. What's wrong with the argument?

② The reflexive, symmetric, + transitive properties do not imply one another. For each possibility, of them being true/false, find a relation with obeying that possibility.

i.e.,

R	S	T	
True	False	False	?
True	True	False	?

Defn Let X be a set with equivalence relation $\sim$.

The equivalence class $[x]$ of element $x \in X$ consists of all elements of X that are equivalent to x.

Frequently, we will discuss the set of equivalence classes, which we denote $X/\sim$.

e.g. shirt color $X = \{\text{shirts}\}$
$X/\sim \; = \{\text{shirt colors}\}$

1.2 Bijections

Sunday, January 22, 2012
10:23 PM

To understand equivalence relations on geometric objects, we often try to construct a map from one to the other.

For example, similar triangles : is there a map from triangle T_1 to triangle T_2 which preserves the angles. If so, the triangles are similar. If not, they're not.

Defn Let X, Y be sets. A map (aka function) $f: X \to Y$ is an **injection** (or **one-to-one**) if distinct pts of X must map to distinct points of Y.

i.e., $\forall a, b \in X$, $a \neq b \Rightarrow f(a) \neq f(b)$

(contrapositive) $f(a) = f(b) \Rightarrow a = b$.

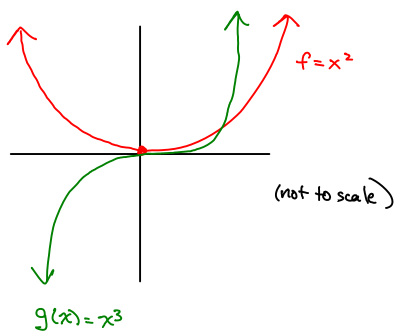
Next, f is a **surjection** (aka **onto**) if $\forall y \in Y$ $\exists x$ mapping to it, i.e., $f(x) = y$.

Finally, f is a **bijection** if it is both an injection and a surjection.

Ex: $f(x) : \mathbb{R} \to [0, \infty)$
$f(x) = x^2$

- not an injection, since $f(-1) = f(1) = 1$

- is a surjection

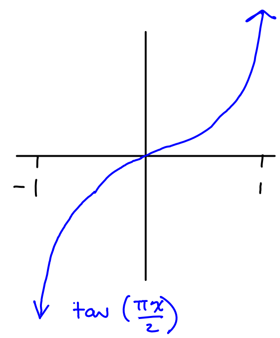


Ex: $g(x) : \mathbb{R} \to \mathbb{R}$
$g(x) = x^3$ is a bijection.

Ex: Find a bijection from $(-1, 1)$ to $\mathbb{R}$.

Recall that $\tan(x)$ maps $(-\pi/2, \pi/2)$ to all of $\mathbb{R}$

Thus $\tan\left(\frac{\pi x}{2}\right)$ maps $(-1, 1)$ to all of $\mathbb{R}$



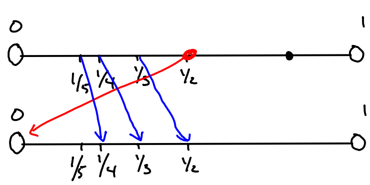
Fact: If a bijection from X to Y exists, then X and Y are the same size.

(DX) check this if either is finite.

Warning: there are different sizes of infinite sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are "countable"
$\mathbb{R}$ is "uncountable"

Ex: Find a bijection from $(0, 1)$ to $[0, 1)$

Idea: something must map to 0; try 1/2

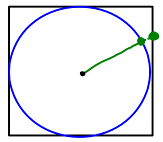


send $\frac{1}{3} \to \frac{1}{2}$
$\frac{1}{4} \to \frac{1}{3}$
$\frac{1}{n} \to \frac{1}{n-1}$ $(n > 2)$

otherwise, send $x \notin \left\{ \frac{1}{n} : n \geq 2 \right\}$ to itself.

This is <u>not</u> continuous.

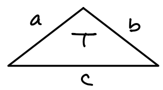
Ex: bijection from unit circle $S^1 = \{(x,y) : x^2+y^2=1\}$ to square formed by $\max(|x|,|y|) = 1$



Idea: push out radially

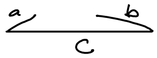
Ex: Find a subset of $\mathbb{R}^3$ and a bijection to the "congruence classes of triangles" — i.e., the equivalence classes of triangles under the (equivalence) relation of congruence.

Given triangle T with side lengths a, b, c.
we may assume $a \le b \le c$.



We claim that any other triangle with these side lengths must be congruent to T. (DX – convince yourself)

Does any triple (a,b,c) produce a triangle?
No — the side lengths must obey the Triangle Inequality.
$c \le a+b$



If $c > a+b$, we cannot form a triangle.

If $c = a+b$, the triangle has degenerated to a line segment.

Consider this set in $\mathbb{R}^3$:
$D = \{(a,b,c) : 0 < a \le b \le c < a+b\}$

Viewing a geometric object as a single point in the space of all such objects leads to the modern idea of configuration spaces.

Research breakthrough (2011) The space of all n-sided polygons of fixed perimeter (say 1) has a natural bijection with the space of all oriented planes in $\mathbb{C}^n$, a well-studied space known as a Stiefel manifold.
(Cantarella, Deguchi, Shonkwiler).

§1.2 Bijections

motivation

injection, surjection

$0,1 \to \mathbb{R}$ $\quad\quad f = \tan x$

$$g = \tan\left(\pi x - \pi/2\right)$$

$(0,1) \to [0,1)$ $\quad$ harder

- $S^1 \to$ square in $\mathbb{R}^2$

Cartesion product $\quad A \times B = \{(a,b) : a \in A, b \in B\}$

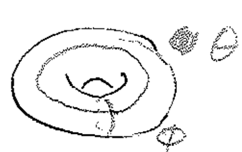
Topologically $(0,1)$ and $[0,1)$ are very different (open/closed)
so bijections don't tell us precisely what we want.
We need something stronger (continuity).

Examples ① subset of $\mathbb{R}^3$ and $\{\text{Triangles}\}/\sim$

[in notes]

where 2 triangles are equivalent if congruent

congruence classes

②  $\longrightarrow S^1 \times S^1$

torus

longitude, meridian

Defns **Composition**

$$g \circ f(x) = g(f(x))$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Identity fcn id_X

inverse fcn

$f : X \to Y$

$g \circ f = id_X$

$g : Y \to X$ $\quad g = f^{-1}$

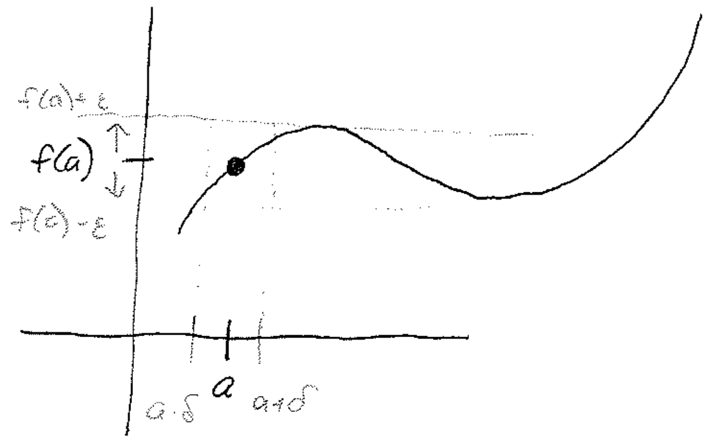
$f \circ g = id_Y$

Theorem $f : X \to Y$ is a bijection $\Leftrightarrow$ f^{-1} exists.

§1.3 Continuity

When is $f: X \to Y$ cts?

In calc I - cts ~ pencil not lifting.



f is cts at a if

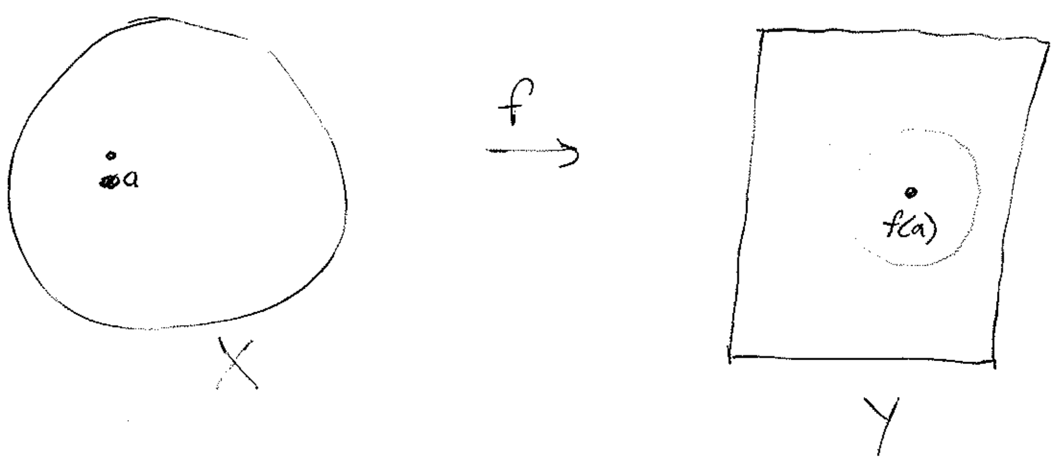
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

if $|x - a| < \delta$

then $|f(x) - f(a)| < \varepsilon$

i.e., points that are close together stay close together.

In general $f: X \to Y$



same defn.

f is <u>continuous</u> at $a \in X$ if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t

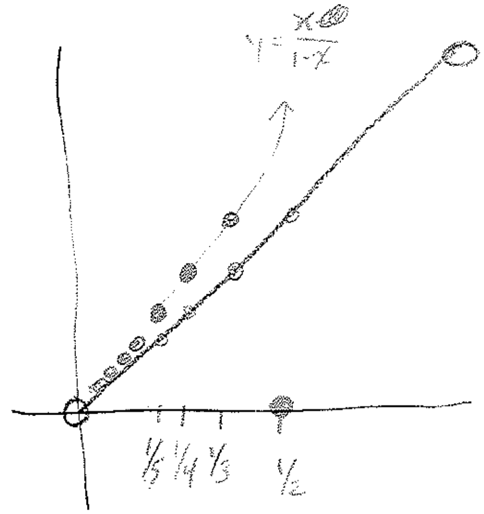
if distance $(x, a) < \delta$ then distance $(f(x), f(a)) < \varepsilon$

metric

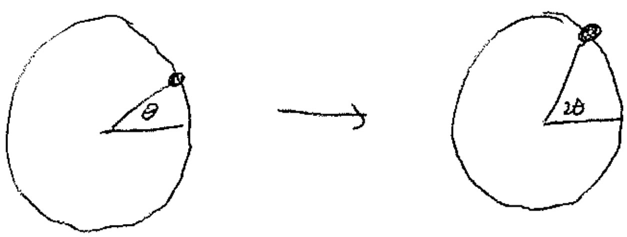
weird fact: $\exists$ non-metrizable spaces (see math 731)

<u>Examples</u> $f: [0,1) \to [0,1)$ is not cts.

discts at each $x = \frac{1}{n}$ $(n \geq 2)$



Ex: $S^1 \to S^1$
unit circle.



$s: z \to z^2$

$x+iy \to$

In polar coords.

$z = re^{i\theta}$ S^1 = unit circle, so $r = 1$

$z = e^{i\theta}$

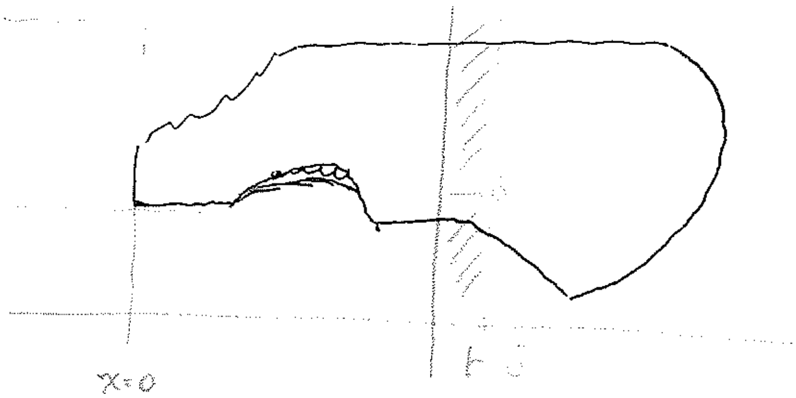
$z^2 = (e^{i\theta})^2 = e^{i2\theta}$

geometrically this is cts.

if we measure distance on S^1 via θ

to make $\underbrace{d(s(z), s(z_0))}_{= d(z,z_0)^2} < \varepsilon$, choose z s.t. $d(z, z_0) < \delta = \varepsilon/2$

Ex: Area function $A(x)$ for planar region R of finite area

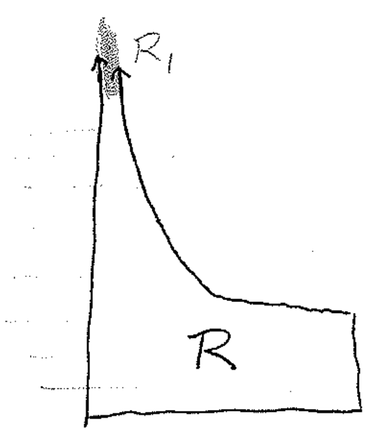


If R is bounded, put $x=0$ on left side and keep R between $y=0$ and $y=b$.

Moving δ in x can add at most $b\cdot\delta$

So, to make $b\delta<\varepsilon$, we want $\delta<\frac{\varepsilon}{b}$.

If R is unbounded in y, we must be more careful.
(If R is bounded in y + unbounded in x, above plan works fine).



Idea: since R has finite area

by drawing $y=\pm1$, $y=\pm2$, $y=\pm3$, ...

we eventually contain most of the area of R.

Given ε, we can draw some $y=b$ such that less than $\varepsilon/2$ of area of R is above $y=b$ or below $y=-b$.
(Actually showing this is a pain.)

Call that region R_1

Then by moving in x, we may add some of R_1 and at most $2b\cdot\delta$

$$A(x)-A(x_0)\le 2b\cdot\delta+\text{Area}(R_1)$$
$$\le 2b\delta+\varepsilon/2$$

want $2b\delta < \varepsilon/2$

$$\delta < \frac{\varepsilon}{4b}. \qquad \square$$

1.4 Topological Equivalence

Monday, January 30, 2012
8:26 AM

Motivation We have talked about equivalence relations — a way of saying that 2 objects are "the same" (with respect to that relation). For example,

- congruence on the set of triangles tells us when triangles are the same size.
- similarity on the set of triangles tells us when triangles have the same angles.
- Bijections on sets tell us when sets have ~~the~~ same size
- in topology, what guarantees that two objects are the same ? i.e., all topological properties are the same

  It should be unsurprising that continuity is important.

Defn A homeomorphism $h: X \to Y$ is a bijection from X to Y with both h and h^{-1} continuous.
We say that X and Y are homeomorphic if there exists a homeomorphism between them.

n.b., to talk about continuity, it seems like we need a notion of distance (i.e., a metric). Sets that have a metric are called metric spaces.

3 Weird Facts

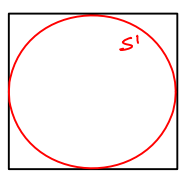
1. Sets may have a lot of different metrics, different enough that the idea of continuity changes.
2. There are sets that do not admit any metrics.

3. On these sets, we may still define continuity (by specifying what open + closed sets are). Once we do so, we call it a topological space. All metric spaces are topological spaces, but not vice-versa.
   We won't work with any of the weird sets in 2, or weird metrics in 1.

Ex: $h: S^1 \to$ square

is a homeomorphism.

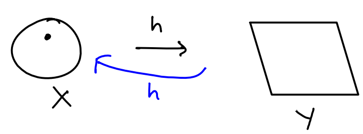
bijection ✓
h cts ✓
h^{-1} cts ✓



Proposition Homeomorphic is an equivalence relation on metric spaces.

Reflexive: Identity map $id: X \to X$ is a homeomorphism.

Symmetric: If $h: X \to Y$ is a homeo., consider $g = h^{-1}: Y \to X$.



h is a bijection $\iff$ $g = h^{-1}$ is a bijection
h is cts. $\iff$ $g^{-1} = (h^{-1})^{-1} = h$ is cts.
h^{-1} is cts. $\iff$ $g = h^{-1}$ is cts.

Thus h is a homeo. $\iff$ h^{-1} is.

Transitive If $f: X \to Y$ and $g: Y \to Z$ are homeos., take the composition $h = g \circ f: X \to Z$.

Composition of bijections is a bijection
Composition of cts maps is cts.

$h^{-1}: Z \to X$
$h^{-1} = (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

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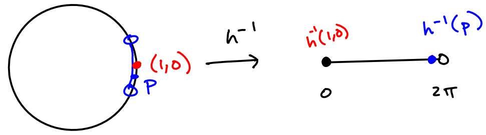
Ex: $\omega: [0, 2\pi) \to S^1$ is a continuous bijection.

$\omega(t) = (\cos t, \sin t)$

However $\omega^{-1}: S^1 \to [0, 2\pi)$ is not a bijection.



The points $(1,0)$ and p can be chosen arbitrarily close together on S^1 but $h^{-1}(1,0) = 0$ while $h^{-1}(p)$ is close to 2π. $\therefore$ h^{-1} is not cts. at $(1,0)$.

$\therefore$ ω is not a homeomorphism.

Topologically, a circle and an interval are fundamentally different. No homeomorphism between them exists.

Some notation

standard disk is $\{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$

unit disk is anything congruent to it.

disk (D^2 or B^2) is anything homeomorphic to it.

standard n-dim. ball is $\{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\} \subset \mathbb{R}^n$

unit n-ball is anything congruent to it

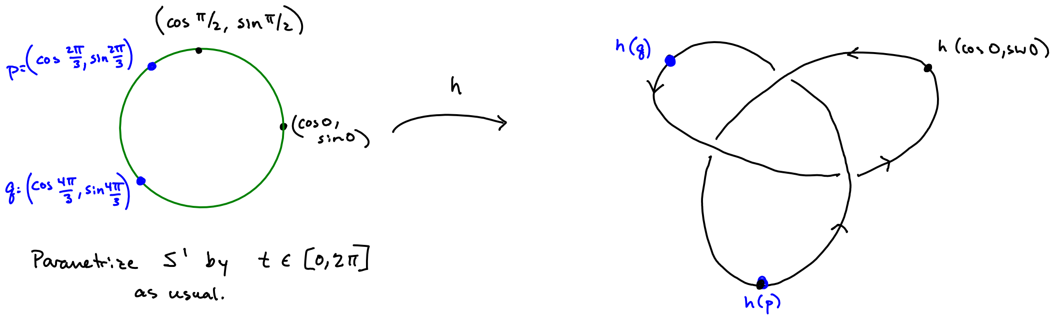
n-ball B^n is any set homeomorphic to it

standard n-dim. sphere is $\{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. It is the boundary of the standard $(n+1)$-ball.

unit n-dim. sphere is anything congruent to it.

n-sphere S^n is anything homeomorphic to it.

Example: Show that a circle S^1 is homeomorphic to the trefoil knot below



Parametrize S^1 by $t \in [0, 2\pi]$ as usual.

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- Since all circles are homeomorphic to the unit circle, we may start with the unit circle.
- h is a bijection

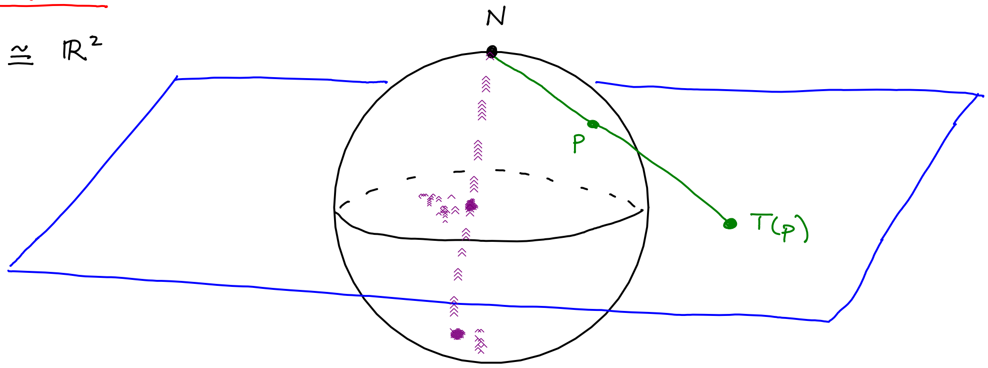
  h is cts : pts on circle that are close, stay close

  h^{-1} is cts : ditto.

Stereographic Projection

Fact $S^2 \setminus \{pt\} \cong \mathbb{R}^2$

Draw $\mathbb{R}^2$ as the equatorial plane.



Define $T: S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$:

T takes a point p on the sphere to a point $T(p)$ — where the line from N through p intersects the plane.

Today: what is $T(s)$? south pole

where does T send the equator ?

the northern hemisphere ?

the southern hemisphere ?

Interpret where T might send N ?

HW show
$$T(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$
$$T^{-1}(u,v,0) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

Hint: use cylindrical coords. instead of (x,y,z) and $(u,v,0)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

1.5 Topological Invariants

Wednesday, February 01, 2012
10:43 AM

Defn a property that is preserved under homeomorphisms is a topological invariant.

Ex: $\{1,2,3\}$ and $\{1,2,3,4\}$ are not homeomorphic.

size/cardinality of a set is preserved via bijections + is ergo a top invt.

Ex: $\mathbb{N}$ and $\mathbb{R}$ are not homeomorphic

Defn a path from a to b in X is a cts. map $\alpha: [0,1] \to X$ with $\alpha(0)=a$, $\alpha(1)=b$.

The path component of a is the set of all points connected to a via a path.

- being in same path compt is equiv reln.

X is path-connected if all points of X lie in the same path component; i.e., if X only has 1 path compt.

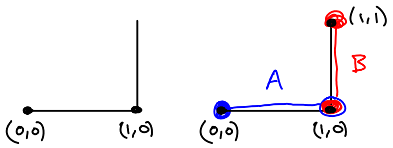
b

Being path connected is a top. invariant.

Theorem 1.44 Suppose $\alpha: [0,1] \to A \cup B$ is a path with $\alpha(0) \in A$, $\alpha(1) \in B$.

Then $\exists$ seq of points in A converging to a point in B or
$\exists$ seq of points in B converging to a point in A.

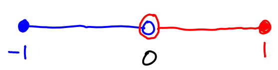
Ex:



Proving this requires some facts from analysis (upper bounds, completeness) — read this, but we don't need it

Ex: $X = [-1,0) \cup (0,1]$ is not path connected

(pf) We can connect all points in $[-1,0)$ to -1. Thus they are all in the same path compt.
We can also connect all pts in $(0,1]$ to 1. Thus they are all in the same path compt



We can use the theorem to show -1 and 1 are not path connected. We will argue by contradiction.
(we assume the opposite of what we want to show.)

What should A be? B?
$A = [-1,0)$
$B = (0,1]$

Suppose -1 and 1 are path connected. Then, there is some path $\alpha: [0,1] \to X = A \cup B$ s.t.
$\alpha(0) = -1$, $\alpha(1) = 1$. Then the theorem implies either

(1) $\exists$ sequence in $A = [-1,0)$ converging to a point in $B = (0,1]$, i.e., to a positive value

(2) $\exists$ sequence in $B = (0,1]$ converging to a point in $A = [-1,0)$, i.e., to a negative value

Q: Take a convergent sequence $(a_n)_{n=1}^{\infty}$ with all terms positive. Must it converge to a positive number?

No. $(\frac{1}{n}) \to 0$. But its limit is either positive or 0; it cannot converge to a negative value.

So (2) cannot happen. Similarly (1) cannot happen.

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Thus, either our theorem is false (it's not!) or our initial assumption must be false.

We may then conclude that there is no path from -1 to 1 in X.

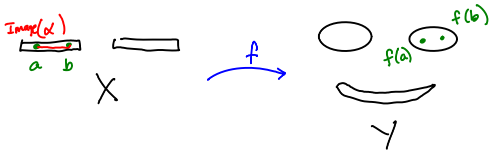
$\therefore$ X has 2 path components.

Ex: Any interval $(a,b) \cup (b,c)$ is not path connected. (DX)

Theorem Let $f: X \to Y$ be continuous. Then f maps a path component of X into some path compt. of Y.

n.b., not necessarily onto

Proof:



Take a, b in some path compt of X.
We must show $f(a), f(b)$ lie in the same path compt of Y.

There is some path $\alpha: [0,1] \to X$ from $\alpha(0) = a$ to $\alpha(1) = b$.

Then consider the composition

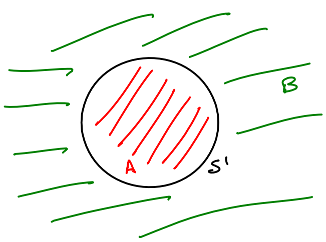
$$f \circ \alpha : [0,1] \to X \to Y$$
$$[0,1] \longrightarrow Y$$

Is it cts? Yes — the composition of cts fcns is cts.

It maps 0 to $f \circ \alpha(0) = f(\alpha(0)) = f(a)$

and 1 to $f \circ \alpha(1) = f(\alpha(1)) = f(b)$. $\square$

Ex: (DX) Determine the path components of $\mathbb{R}^2 \setminus S^1$.



Answer: 2 path components: (A) inner disk + (B) points outside

To show it,

(1) find a path connecting any 2 points of A

— use a straight line

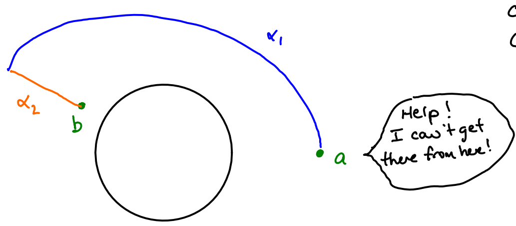
(2) find a path connecting any 2 points of B

— can we always use a straight line? No.

If we cannot, first travel around a concentric circle to S^1 containing a, until you reach the polar angle of b.

Call this path $\alpha_1 : [0, 1/2] \to B$

Then if necessary move radially to reach b — call this $\alpha_2 : [1/2, 1] \to B$.



We form α as $\alpha(t) = \begin{cases} \alpha_1(t) & t \in [0, 1/2] \\ \alpha_2(t) & t \in [1/2, 1] \end{cases}$.

Defn A subset Y of a path-connected set X **separates** X if $X - Y$ is not path-connected

Above, S^1 separates $\mathbb{R}^2$.

- The general version of this example is much harder ... that any curve homeomorphic to S^1 separates $\mathbb{R}^2$. It's known as the Jordan Curve Theorem, named after Camille Jordan (French, 1887). His proof was incomplete ... full proof by Veblen (Princeton, 1913).

Notation C_a = path-compt of a in X

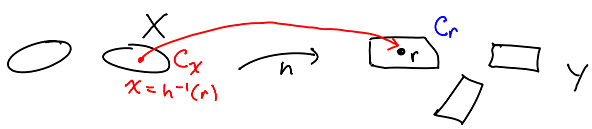
$P(X) = \{$path compts of $X\}$

Corollary Any homeomorphism $h: X \to Y$ induces a bijection $h_*: P(X) \to P(Y)$ on the path components. In particular, homeomorphic spaces must have the same number of path components.

Proof: First, h_* is a well-defined map, since by the theorem, all pts of path compt C_x of X land in the path compt $C_{h(x)}$ of Y. So $h_*(C_x) = C_{h(x)}$.

We show h_* is a bijection:

- h_* is a surjection: Consider a path component of Y containing r, i.e., path compt C_r. Since h is onto, $\exists\ x$ s.t. $h(x) = r$. By the theorem, h must map all points of C_x into C_r.



Thus $C_r = h_*(C_x)$.

$\therefore$ h_* is onto.

- h_* is an injection: For $x, z \in X$, suppose C_x and C_z both map via h_* to C_r, i.e., $h_*(C_x) = h_*(C_z) = C_r$. Consider h^{-1} which is cts since h is a homeo.
Since $h^{-1}(h(x)) = x \in C_x$, h^{-1} maps C_r into C_x.
Since $h^{-1}(h(z)) = z \in C_z$, h^{-1} maps C_r into C_z.

By the theorem C_x and C_z must be the same path compt.

$\therefore$ h_* is 1-1, and thus it's a bijection.

Since bijections preserve sizes of sets, $P(X)$ and $P(Y)$ have the same cardinality, i.e., X and Y have the same number of path compts.

Example: (important!) Show that $(0,1)$ and $[0,1)$ are not homeomorphic.

We found a bijection between them in §1.2, but it was not continuous! But this means they have the same cardinality, so we cannot use that invariant.

Lets try path-connectedness. Are both path-connected? Yes.

We only know a few invariants. Should we give up? No! Let's use separability into path compts.

Removing the point $0 \in [0,1)$ does not separate it.

If $h: [0,1) \to (0,1)$ were a homeo., this means $h(0)$ should not separate $(0,1)$.

Q: which points of $(0,1)$ separate it?

$\to$ All of them. So $h(0)$ does not separate $(0,1)$ for any map $h: [0,1) \to (0,1)$

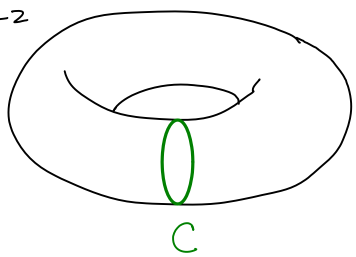
$\therefore$ No such h is a homeomorphism

$\therefore$ $[0,1) \not\cong (0,1)$

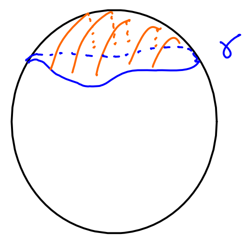
Example: $T^2 \not\cong S^2$

Draw a closed curve on S^2 — it separates S^2. (DX)

torus T^2



C does not separate T^2



γ separates S^2

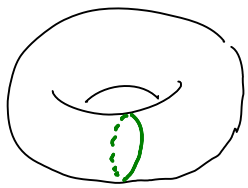
Since there are closed curves on the torus that separate it, while no such curves exist on the sphere, we see the torus and sphere are not homeomorphic.

1.6 Isotopy

Wednesday, February 08, 2012
9:46 AM

Not all homeomorphisms $h: X \to Y$ can be obtained by deforming X:

Example



Defn Let A, B be subsets of X. An ambient isotopy in X from A to B is a map $h: X \times [0,1] \to X$ such that, if $h(x,t)$ is denoted $h_t(x)$,

(1) $h_t : X \to X$ is a homeomorphism for each $t \in [0,1]$

(2) $h_0(x)$ is the identity map

(3) $h_1(A) = B$, i.e., h_1 sends A to B

By the end of the isotopy, A has been deformed to B.

§1.6 Isotopy

For Fri.

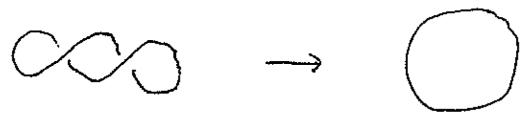
Come up with 3 spaces that are not homeo.
And 2 more spaces that are homeo to your first one.

homeos vs deformations

Ex: Dehn twist

Ex1: Deform any triangle to any other.

Ex2:



ambient isotopy

what was important for triangles?
for knots?

$h: X$
$A \to B$

- we start at $A \subset X$
- we move through time by changing X slightly
- we end at $B \subset X$
- at every time, did we still have a triangle? yes
  an unknot? yes. important

we can think of time as going from $0..1$

at each time t, we have a homeo $h_t : X \to X$

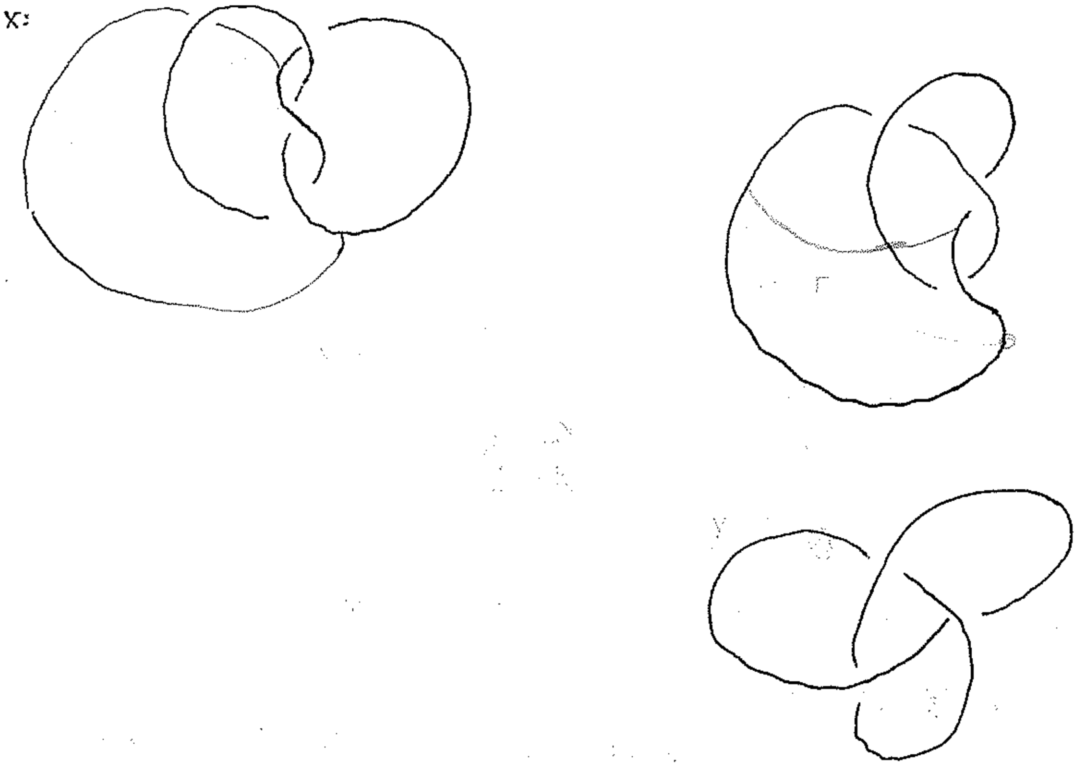
$h_0 = id$

h_1 sends A to B

Since we move by changing X slightly,
we ask that h_t be cts in X (it's a homeomorphism)

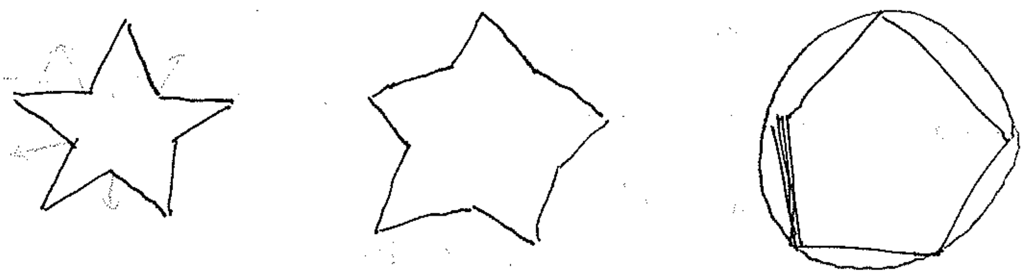
and in t:

Ex:



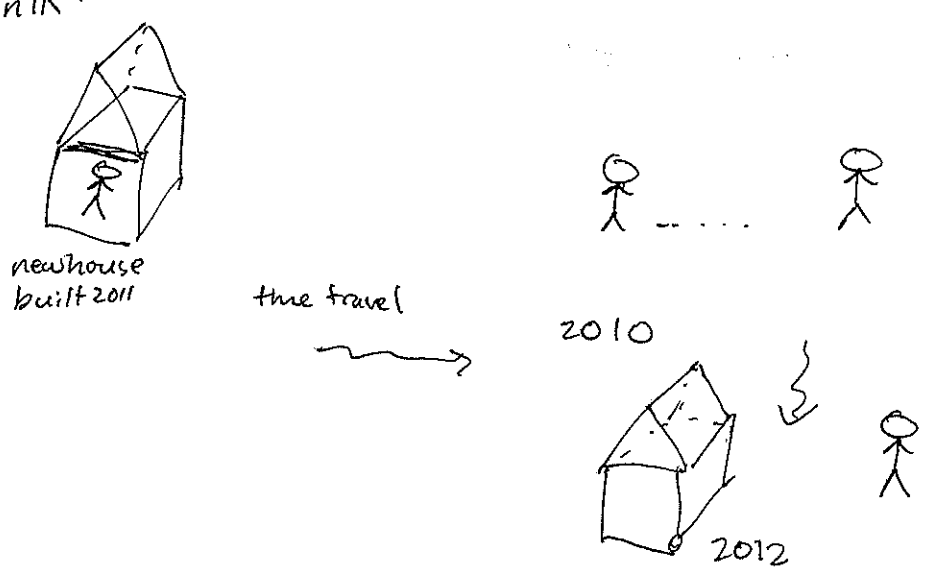
trefoil

Ex:



now make round

Ex: knots in $\mathbb{R}^4$



Ex: