Interpreting mathematical shapes through motion

Monday, 17 September 2007

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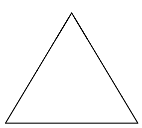
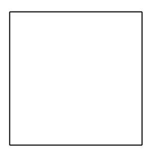
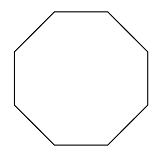
Today we consider a few important mathematical shapes and allow you to interpret them. We'll start with Platonic Solids.

1 Platonic Solids

Regular polygons

In two dimensions, one can form a *equilateral triangle* by using three sides of the same length (and three equal angles). One can also form a *square*: four equal sides, four equal angles. And a *regular pentagon*: five equal sides, five equal angles. These are all called **regular polygons**. A stop sign is a red *regular octagon*.

For any number of sides N bigger than 2, one can make a *regular N-gon*, with N equal sides and N equal angles.

1.1 Regular shapes in 3 dimensions

Question: So what's the analog of a square in three dimensions?

Answer: A cube! It has six faces (all squares). All of its angles & sides are the same, and all corners look the same.

The analog of an equilateral triangle is a triangular pyramid, known as a **tetrahedron**. Like the cube, all of its angles & sides are the same, and all corners look the same. It has four faces (all triangles). The cube and tetrahedron are examples of **Platonic solids**, the 3-d analog of regular polygons.

So what is the missing Platonic solid with five faces? *There isn't one!* (not with equal sides, angles, and corners)

There are only five Platonic solids in existence. See their pictures on the last page.

- tetrahedron: 4 faces (triangles)
- cube: 6 faces (squares)
- octahedron: 8 faces (triangles)
- dodecahedron: 12 faces (pentagons)
- icosahedron: 20 faces (triangles)

Why are there only 5 Platonic solids? Ask one of the math 107 students who are participating today. It has to do with the angles at a corner adding up to less than 360°.

1.2 Duality

One Platonic solid sits in another as follows: take each face and draw a point at the center of the face. These become the corners of the inside solid. Draw an edge between two new corners if they faces of the original solid meet along an edge. When one Platonic solid sits inside another, they are said to be **dual solids**. The 5 solids like to pair up with each other:

- A cube is dual to a octahedron and vice-versa.
- A dodecahedron is dual to a icosahedron and vice-versa.

Oh no! the poor tetrahedron is left out; what should it pair with? Itself!

- A tetrahedron is dual to a another tetrahedron

Start with a cube, it has an octahedron inside, which has a smaller cube inside it, which has a smaller octahedron inside it, which has a smaller cube inside it, which has . . . And the original cube is in a larger octahedron, which is in a larger cube, which is in a larger octahedron . . .

How could you interpret duality through movement? What ideas do you have regarding the patterns of dual solids which get smaller & smaller? Larger & larger?

1.3 Interpret Platonic Solids through motion

- In appropriately sized groups, form a tetrahedron and a cube. Try forming each shape from different perspectives. (e.g., what if the tetrahedron is resting on a face? on a point?) How does this affect your movements?

- We will then reorganize the groups and have you form an octahedron, a dodecahedron, and an icosahedron. How does the symmetry of these objects manifest in your motions?

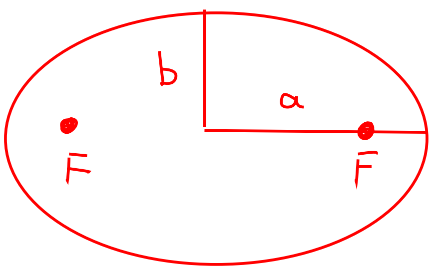
- As a whole class, interpret the duality of a cube and an octahedron. Are you drawn towards having the cube lie on the inside or outside of the octahedron?

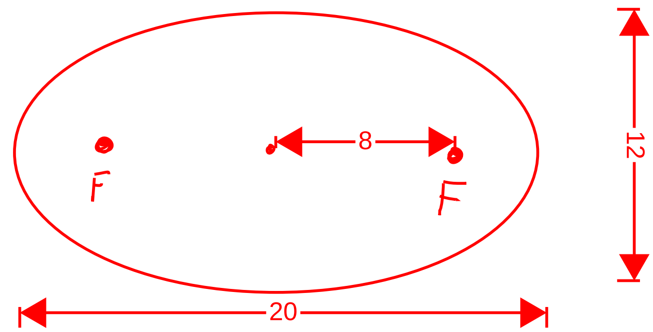
- If time permits, interpret the duality of a dodecahedron with an icosahedron.

2 Ellipses

An **ellipse** occurs when a circle is stretched in one direction. We'll assume it's stretched horizontally as below. On the next page is a drawing of an ellipse in general, and one specific example that is 20 units long and 12 units high (in the formula below, this means $a = 10$ and $b = 6$).

The **foci** (singular: focus) are two points symmetrically placed in the ellipse; they lie along the long axis, each $\sqrt{a^2 - b^2}$ (8 units in our example) from the center.





Formula: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Our example: $\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$

2.1 Constructing an ellipse

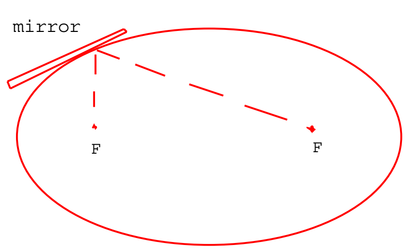
- Measure and cut $20ft. = 2a$ of rope.
- Place two people $16ft.$ apart; they are the *foci* of our ellipse.
- Each of these foci should be holding one end of the rope.
- A third person should pull the rope taut while our foci hold it.
- The third person & others should walk around these two foci, keeping the rope taut at all times. The path traced out is our ellipse.

Stated mathematically, an ellipse consists of those points *whose distances to the two foci add up to a constant.* The bigger the constant, the bigger the ellipse.

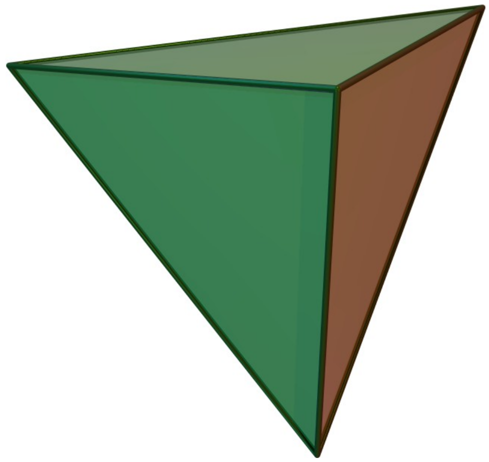
2.2 Reflections & ellipses

Question: Say we place a mirror Along our ellipse, tangent to it. What does one of our foci see in the mirror?
Answer: The other focus.

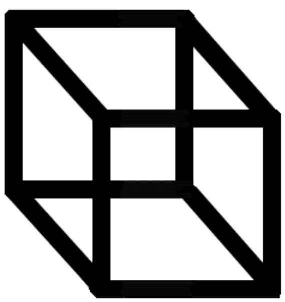
Try it. First start with the mirror along the smaller axis of the ellipse as shown below. Then try moving the mirror to any other point on the curve.



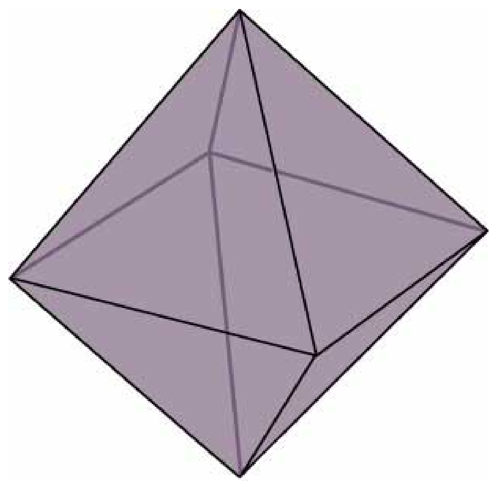
3 The 5 Platonic Solids



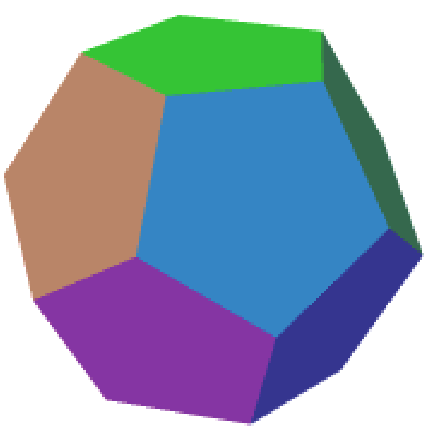
Tetrahedron (4 faces)



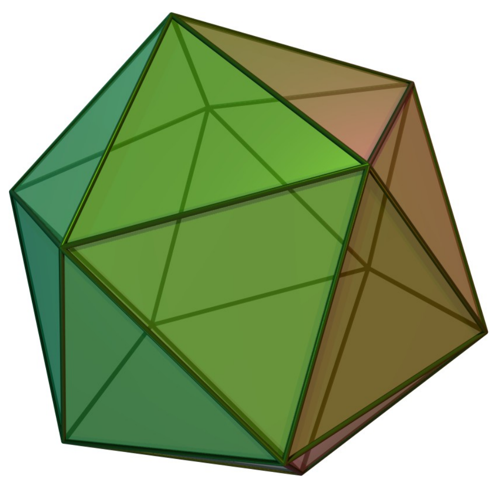
Cube (6 faces)



Octahedron (8 faces)



Dodecahedron (12 faces)



Icosahedron (20 faces)