crystals. The fluoride cubic crystals with perovskite structure seem to be advantageous for that purpose. It is evident that both ionic and covalent contributions to the zero-field splitting are smaller in fluorides than in oxides. But from our calculations it appears that the covalent contribution to the ratio G_{11}/G_{44} should be different for fluorides than for oxides. From the considerations given above, the ratio G_{11}/G_{44} should be greater for fluorides than for oxides because of more contracted fluorine $2p$ wave functions.

Note added in proof. Recently the values of G_{11} and G_{44} for Cr^{3+} and Ni^{2+} in MgO were computed by Tucker[22,23] using the point-charge model. These values differ greatly from the point-charge contributions estimated in this paper for the following reasons: (i) Tucker's values are obtained from a fit to the experimental value of the cubic field splitting[22]; thus they are three or four times greater than the values computed from the first principles. (ii) The expression for G_{44} used by Tucker is smaller by a factor of 3 than in our expressions (10) or (17), in which the terms caused by the excited states t_2^3, ${}^2T_2$ or $t_2^5e^3$, ${}^1T_2$ are truncated and near-neighbor model is used. (iii) The effect of further neighbors is not negligible, and some terms differ by as much as 50% when Kanamori sums are used instead of the near-neighbor model. (iv) The effect of the excited states t_2^3, ${}^2T_2$ and/or $t_2^5e^3$, ${}^1T_2$ was not considered in Altshuler *et al.*'s[24] expression for G_{44} used by Tucker, though their effect is substantial.

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The author is indebted to Dr. Z. Šroubek and Dr. E. Šimánek for their continuous interest in this work.

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Quantum Theory of an Optical Maser.* I. General Theory

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A quantum statistical analysis of an optical maser is presented in generalization of the recent semiclassical theory of Lamb. Equations of motion for the density matrix of the quantized electromagnetic field are derived. These equations describe the irreversible dynamics of the laser radiation in all regions of operation (above, below, and at threshold). Nonlinearities play an essential role in this problem. The diagonal equations of motion for the radiation are found to have an apparent physical interpretation. At steady state, these equations may be solved via detailed-balance considerations to yield the photon statistical distribution $\rho_{n,n}$. The resulting distribution has a variance which is significantly larger than that for coherent light. The off-diagonal elements of the radiation density matrix describe the effects of phase diffusion in general and provide the spectral profile $|E(\omega)|^2$ as a special case. A detailed discussion of the physics involved in this paper is given in the concluding sections. The theory of the laser adds another example to the short list of solved problems in irreversible quantum statistical mechanics.

I. INTRODUCTION

THE theory of an optical maser due to Lamb[1] treats the atoms quantum-mechanically while considering the radiation as a classical electromagnetic field. This theory has provided a basis for understanding a wide range of observed laser phenomena and has been extensively tested by Javan and Szöke,[2] Fork and Pollack,[3] and others. Extensions of the theory to allow for the presence of a magnetic field[4,5] or cavity anisotropy[6] have been made by several authors, and there is no doubt that remarkable fits are being obtained with experimental data. The ring laser has been analyzed by Aronowitz,[7] and by Gyorffy and Lamb,[8] again in good agreement with observations. Various forms of modulation can be discussed, as in the work of Harris.[9] The buildup in time of oscillations from a

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† This paper is based on a thesis submitted by M. Scully to Yale University in partial fulfillment of the requirements for the Ph.D degree.
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very low level has been investigated by Pariser and Marshall,[10] and satisfactory accord with theory is obtained.

In view of the successes of the semiclassical theory, it may be asked why there is need for a better treatment. One reason is that the foregoing theory implies that laser radiation in an ideal steady state is absolutely monochromatic. To be sure, an actual laser has mechanical and statistical disturbances, and these give rise to a finite radiation band width. The intrinsic line width, expressing the effects of thermal noise, vacuum fluctuation fields, and spontaneous emission, is in any case far too small to detect with present techniques. Still, the proper calculation of such effects has provided a challenging problem in nonequilibrium statistical mechanics. Another defect of the semiclassical theory is that oscillations will not grow spontaneously, but require an initial optical-frequency (o.f.) field from which to start. One would like to know how oscillations can develop from a state with no radiation initially present. Since spontaneous radiation must be involved, it is clear that this kind of question requires the quantum theory of radiation.

Still another problem requiring a fully quantum-mechanical theory is to determine the statistical distribution of the energy stored in the laser cavity, i.e., the "photon" statistics. This information is a prerequisite for a proper discussion of the statistical distribution of photoelectrons[11–14] produced by a laser.

A number of papers have appeared recently dealing with a quantum-mechanical laser. The earliest of these replaced the photon emission and annihilation operators by c numbers,[15] or prematurely factored[16] the density matrix, and hence are a disguised form of the semiclassical theory. Extensions of the semiclassical theory to include an injected noise signal have been given.[17–19]

We now turn to an enumeration of the fully quantum-mechanical treatments. One of these has been given by the authors[20] and extended in a recent publication.[21] The present paper is a detailed account of that theory and is the first in a series on the quantum theory of the laser. The treatment will closely parallel that of the semiclassical theory. McCumber,[22] Kemmeny,[23] and Korenman[24] have applied a Green's-function technique to the problem. Lax[25] has also given a treatment of the laser spectrum by postulating quantum noise sources determined from general considerations, and in collaboration with Louisell,[26] has subsequently calculated an equation of motion for the density matrix. Willis[27] has extended his earlier treatment, based on methods due to Bogoliubov. The approach of the Haken school has been generalized[28] to include quantum noise sources. The recent results of Fleck[29] are similar to those presented in Ref. 20.

Before developing the quantum theory, it is desirable to review briefly the semiclassical theory. We are interested in the electromagnetic field in a cavity resonator which for optical frequencies can consist of two plane-parallel mirrors. In the semiclassical theory,[1] it was assumed that a known electromagnetic field $E(z,t)$ was present, which consisted of one or more superposed normal modes of oscillation of the cavity as given by

$$E(z,t)=\sum_n E_n(t)\cos[\nu_n t+\varphi_n(t)]\sin(n\pi z/L). \quad (1)$$

The spatial dependence of the normal modes was taken to be as simple as possible. Each mode of the electromagnetic field was specified by an amplitude $E_n(t)$ and a phase angle $\varphi_n(t)$, which were regarded as slowly varying in an optical period. The frequency of each term was denoted by ν_n. The wave equation for the electric field with a driving force term on the right-hand side involving the electric polarization of the medium $P(t)$, and an Ohmic dissipation proportional to a fictitious conductivity σ, was

$$\mu_0\epsilon_0\partial^2\mathbf{E}/\partial t^2+\mu_0\sigma\partial\mathbf{E}/\partial t+\nabla\times(\nabla\times\mathbf{E})=-\mu_0\partial^2\mathbf{P}/\partial t^2. \quad (2)$$

In the solution of the inhomogeneous wave equation the projection on the cavity modes P_n of the electric polarization $P(z,t)$, and their in-phase and out-of-phase amplitudes $C_n(t)$ and $S_n(t)$ played an important role. One had the relation

$$P_n(t)=C_n(t)\cos\{\nu_n t+\varphi_n(t)\}+S_n(t)\sin\{\nu_n t+\varphi_n(t)\}. \quad (3)$$

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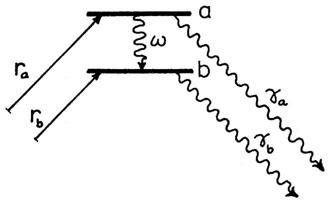

FIG. 1. Maser action takes place between the two excited energy levels a and b separated by a frequency $\omega>0$. These levels are excited at rates r_a and r_b, while the atomic decay constants are given by γ_a and γ_b.

The self-consistency approximation involves calculation of the polarization, i.e., C_n and S_n, of the active medium on the assumption that the electric field E is known, and then substituting that polarization in the right-hand side of the wave equation (2), requiring that the polarization should produce the field which was initially assumed. The result of this requirement is a pair of equations, giving the amplitudes $E_n(t)$ and the frequencies ν_n or phases $\varphi_n(t)$ of each mode of the radiation field:

$$(\nu_n+\dot{\varphi}_n-\Omega_n)E_n=-\tfrac{1}{2}(\nu/\epsilon_0)C_n, \tag{4a}$$

$$\dot{E}_n+\tfrac{1}{2}(\nu/Q_n)E_n=-\tfrac{1}{2}(\nu/\epsilon_0)S_n, \tag{4b}$$

where $\Omega_n=\pi nc/L$ is the cavity resonance frequency and $Q_n=\nu_n\epsilon_0/\sigma_n$ gives the quality factor of the mode.

The next task is to determine the macroscopic driving polarization which is a statistical summation over the microscopic atomic dipoles. The atoms of the active medium are taken to have two excited levels, a and b, separated by a transition frequency ω between which the laser activity is taking place, as in Fig. 1. The levels decay to lower states by radiative decay at rates indicated by γ_a and γ_b. Atoms are excited to these levels by some process such as electron collision from the ground state. Let us imagine that at a time t_0 an atom is brought into state $|a\rangle$ at some point z_0 in the laser cavity. Initially its wave function is ψ_a, but because of the presence of the assumed optical-frequency field, the atom's wave function becomes a linear combination of energy eigenstates, ψ_a and ψ_b. Instead of using wave functions, it is better to work with the elements of a density matrix ρ, which is a more convenient procedure when one wishes to describe a variety of situations in one formalism. The diagonal elements of the 2×2 density matrix are the probabilities of finding the states a and b occupied while the off-diagonal elements are related to the (quantum-mechanically averaged) induced atomic electric-dipole moment at time t of the atom excited at time t_0:

$$p(t,t_0)=\wp[\rho_{a,b}(t,t_0)+\rho_{b,a}(t,t_0)], \tag{5}$$

where $\wp=ex_{ab}$ is the (real) matrix element of the electric-dipole operator connecting states $|a\rangle$ and $|b\rangle$.

The density matrix obeys the differential equation

$$\dot{\rho}=-i[H,\rho]-\tfrac{1}{2}[\Gamma\rho+\rho\Gamma], \tag{6}$$

where

$$\rho=\begin{pmatrix} a^*a & b^*a \\ a^*b & b^*b \end{pmatrix}, \tag{7}$$

$$H=\begin{pmatrix} W_a & V(t) \\ V(t) & W_b \end{pmatrix}, \tag{8}$$

$$\Gamma=\begin{pmatrix} \gamma_a & 0 \\ 0 & \gamma_b \end{pmatrix}, \tag{9}$$

and for the case of stationary atoms

$$\hbar V(t)=-\wp\sum_{n=1}^{M}E_n(t)\sin\left(\frac{n\pi z}{L}\right)\cos[\nu_n t+\varphi_n(t)]. \tag{10}$$

We are interested in a solution $\rho(t,t_0)$ which satisfies a particular initial condition for $t=t_0$, such as

$$\rho(t_0)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

Because of the perturbation $V(t)$, the atom injected at t_0 acquires an electric-dipole moment which decays with a time constant $1/\gamma_{ab}=2/(\gamma_a+\gamma_b)$. In order to calculate adequately the electric polarization, it is necessary to compute the off-diagonal elements of the density matrix to at least third order in the radiative interaction. Having solved for $P(t,t_0)$, we perform a statistical sum over atoms by integration over entrance times t_0.

The differential equation which determines the amplitude E_n as a function of time when only a single mode can oscillate was found to be

$$\dot{E}_n=\alpha_nE_n-\beta_nE_n^3, \tag{12}$$

where the coefficient α_n was given[30] by

$$\alpha_n=-\tfrac{1}{2}(\nu/Q_n)+\tfrac{1}{2}(\nu\wp^2\bar{N}\gamma_{ab}/\hbar\epsilon_0)[(\omega-\nu_n)^2+\gamma_{ab}^2]^{-1} \tag{13}$$

and is the sum of a negative loss term corresponding to the cavity Q and a positive term characterizing the linear pumping. The latter depends on the number density $N(z)$ of excited atoms only through the excitation $\bar{N}$ defined by

$$\bar{N}=\int_0^L N(z)\sin^2\left(\frac{n\pi z}{L}\right)dz\bigg/\int_0^L\sin^2\left(\frac{n\pi z}{L}\right)dz. \tag{14}$$

The parameter β_n is a measure of atomic saturation, which introduces nonlinearities into the problem, and is given[30] by

$$\beta_n=(\tfrac{3}{8}\wp^4\gamma_{ab}{}^3\nu\bar{N}/\hbar^3\epsilon_0\gamma_a\gamma_b)[(\omega-\nu_n)^2+\gamma_{ab}^2]^{-2}. \tag{15}$$

[30] W. E. Lamb, Jr., in *Proceedings of the International School of Physics "Enrico Fermi," Course XXXI*, edited by P. A. Miles (Academic Press Inc., New York, 1964), p. 92, Eqs. (85) and (86).

The steady-state solution of Eq. (12) is clearly

$$E_n^2=\frac{\alpha_n}{\beta_n}=\frac{[(\text{linear pumping})-(\text{damping})]}{(\text{nonlinear parameter})}. \quad (16a)$$

Equations (12) and (16a) are basic results of the semiclassical analysis and must, by the correspondence principle, have counterparts in a quantum-mechanical theory of a laser operating in the usual region where huge quantum numbers are involved.

Having set the goal as a quantum treatment paralleling the semiclassical theory, we outline electromagnetic-field quantization in Sec. II, present the model and obtain the equation of motion for the radiation density matrix in Sec. III. In Sec. IV we obtain the steady-state photon statistics $\rho_{n,n}$, while the linewidth analysis is included in Sec. V. Discussion of the physics involved in the paper and a summary will be found in Secs. VI and VII.

II. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

A. Quantum Theory of Radiation

In this section we quantize the radiation field corresponding to a typical laser mode, i.e., a scalar field in a finite one-dimensional cavity. Although there are many textbooks which develop the quantum theory of radiation,[31] they treat the problem for an unbounded region and make use of the vector potential. We are here primarily interested in treating the interaction between the laser radiation and decaying atoms in the electric-dipole approximation. It is unnecessary and risky[32,33] to discuss such a problem using the vector potential, and therefore we prefer to develop the quantum theory of radiation in a form more appropriate for quantum electronics, emphasizing the electric and magnetic fields. Maxwell's equations for a classical free field are

$$\nabla\times\mathbf{H}=\partial\mathbf{D}/\partial t, \quad (16b)$$

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, \quad (16c)$$

$$\nabla\cdot\mathbf{B}=0, \quad (16d)$$

$$\nabla\cdot\mathbf{E}=0, \quad (16e)$$

$$\mathbf{B}=\mu_0\mathbf{H}, \qquad \mathbf{D}=\epsilon_0\mathbf{E}. \quad (16f)$$

We take the electric field to be in the x direction and expand in the normal modes of the cavity with an appropriate weighting factor

$$E_x=\sum_s q_s[2\Omega_s^2M_s/(LA\epsilon_0)]^{1/2}\sin(K_sz), \quad (17)$$

where q_s is the normal mode amplitude with the dimensions of a length, $K_s=s\pi/L$, with $s=1, 2, 3, \cdots$, and $\Omega_s=s\pi c/L$ the cavity eigenfrequency. The effective transverse area of the optical resonator is denoted by A. The magnetic field in the cavity as implied by Eqs. (17) and (16b) is

$$H_y=\sum_s(\dot{q}_s\epsilon_0/K_s)[2\Omega_s^2M_s/(LA\epsilon_0)]^{1/2}\cos(K_sz). \quad (18)$$

As is well known, there is an analogy between the dynamical problem of a single mode of the electromagnetic field and that of a mechanical simple harmonic oscillator. We have inserted a quantity M_s into Eqs. (17) and (18) which has the dimensions of a mass in order to emphasize this analogy. The equivalent mechanical oscillator will have a mass M_s and a Cartesian coordinate q_s.

The Hamiltonian for the field

$$H=\tfrac{1}{2}\int d\tau(\epsilon_0E^2+\mu_0H^2) \quad (19)$$

expressed in terms of Eqs. (17) and (18) for E and H becomes

$$H=\tfrac{1}{2}\sum_s[M_s\Omega_s^2q_s^2+M_s\dot{q}_s^2], \quad (20)$$

$$H=\tfrac{1}{2}\sum_s[M_s\Omega_s^2q_s^2+p_s^2/M_s], \quad (21)$$

where $p_s=M_s\dot{q}_s$ is the canonical momentum of the sth mode. Equation (21) expresses the Hamiltonian for the radiation field as a sum of independent oscillator energies. Each mode of the field is dynamically equivalent to a mechanical harmonic oscillator which is quantized by simply taking over the well-known quantization of the mechanical oscillator

$$[p_s, q_{s'}]=(\hbar/i)\delta_{s,s'}, \quad (22a)$$

$$[q_s, q_{s'}]=[p_s, p_{s'}]=0. \quad (22b)$$

The nth stationary-state energy of the sth mode of the field is given by

$$E_n=\hbar\Omega_s(n+\tfrac{1}{2}), \quad (23)$$

and the corresponding wave function is[34]

$$\varphi_n(q_s)=(\alpha/\pi^{1/2}2^nn!)^{1/2}H_n(\alpha q_s)\exp(-\tfrac{1}{2}\alpha^2q_s^2), \quad (24)$$

where $\alpha^2=(M_s\Omega_s/\hbar)$. It is sometimes convenient to make a canonical transformation to operators a_s and $a_s^\dagger$:

$$a_s=[2M_s\hbar\Omega_s]^{-1/2}(M_s\Omega_sq_s+ip_s), \quad (25a)$$

$$a_s^\dagger=[2M_s\hbar\Omega_s]^{-1/2}(M_s\Omega_sq_s-ip_s). \quad (25b)$$

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[31] See, for example, W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1956), 3rd ed.; or W. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Company, Inc., New York, 1965).

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The Hamiltonian and commutation relations implied are

$$H=\hbar\sum_s(a_s^\dagger a_s+\tfrac{1}{2})\Omega_s, \tag{26}$$

$$[a_s, a_{s'}^\dagger]=\delta_{s,s'}, \tag{27a}$$

$$[a_s, a_{s'}]=[a_s^\dagger, a_{s'}^\dagger]=0. \tag{27b}$$

These operators a_s and $a_s^\dagger$ are the usual annihilation and creation operators for the number states of the sth mode of the radiation field

$$a_s \mid n_s\rangle = n_s^{1/2} \mid n_s-1\rangle, \tag{28a}$$

$$a_s^\dagger \mid n_s\rangle = (n_s+1)^{1/2} \mid n_s+1\rangle. \tag{28b}$$

In terms of these operators, the electric field is

$$E_x=\sum_s \mathcal{E}_s(a_s+a_s^\dagger)\sin K_s z, \tag{29}$$

where the quantity

$$\mathcal{E}_s=[\hbar\Omega_s/(LA\epsilon_0)]^{1/2} \tag{30}$$

has the dimension of an electric field.

The states $\mid n\rangle$ are eigenstates of the number operator $a^\dagger a$,

$$a^\dagger a \mid n\rangle = n \mid n\rangle \tag{31}$$

and describe a cavity mode containing exactly n photons. These states have zero average electric field and a mean-square average of

$$\langle[E_x(z)]^2\rangle = \langle n \mid \mathcal{E}_s^2(a^\dagger+a)^2 \mid n\rangle \sin^2 K_s z \tag{32}$$

$$=2\mathcal{E}_s^2(n+\tfrac{1}{2})\sin^2 K_s z. \tag{33}$$

It is the purpose of the next section to investigate more general states of the radiation field, and the electric field calculated from these states. We will be particularly interested in states corresponding to the classical limit of the quantized field.

B. Wave Packets for the Radiation Field

Since the radiation field for a single-cavity mode is dynamically equivalent to the problem of a simple harmonic oscillator, the wave function describing the radiation in the cavity is a linear combination of products of these pure photon eigenstates. For such a state, there would be no definite photon number, but only a distribution of probabilities for finding various numbers of photons if one made an observation of the energy in the cavity. This general state vector for the field is

$$\mid\psi\rangle = \sum_{\{n(s)\}} a_{\{n(s)\}} \mid \{n(s)\}\rangle, \tag{34}$$

where

$$\{n(s)\}=n_1, n_2, \cdots, n_s, \cdots.$$

Concentrating on a single mode of the free field in the q representation,

$$\psi(q,t)=\sum_n a_n(t)\varphi_n(q), \tag{35}$$

$$\psi(q,t)=\sum_n a_n(0)\exp(-in\Omega t)\varphi_n(q). \tag{36}$$

This wave packet, in general, has a nonzero average electric field

$$\langle E(z,t)\rangle=\sin(Kz)\sum_n[a_n^* a_{n+1}(n+1)^{1/2}e^{-i\Omega t}+\mathrm{cc}], \tag{37}$$

which has the sinusoidal spatial dependence of a normal mode and a monochromatic time dependence with frequency Ω.

The photon probability distribution is given by $\mid a_n(0)\mid^2$, and the mean photon number is

$$\langle a^\dagger a\rangle=\sum_n n a_n^* a_n. \tag{38}$$

The probability amplitudes $a_n(0)$ may be determined from the initial form of the wave function $\psi(q, 0)$ by

$$a_n(0)=\int dq_0 \varphi_n(q_0)^*\psi(q_0,0). \tag{39}$$

We may write the wave function at time t as

$$\psi(q,t)=\int dq_0 G(q,q_0,t)\psi(q_0,0), \tag{40}$$

i.e., if at time $t=0$, $\psi=\psi(q,0)$, then the time evolution will be given by folding $\psi(q, 0)$ with the Green's function $G(q, q_0, t)$:

$$G(q,q_0,t)=\sum_n \varphi_n(q_0)^*\varphi_n(q)e^{-in\Omega t}. \tag{41}$$

The physical interpretation of $G(q, q_0, t)$ is that it represents the time development of a wave function which is initially localized as a delta function of q_0. Kennard[35] has given a very ingenious derivation of G, based on the observation that the Green's function is an eigenfunction of the Heisenberg operator $q(-t)$ having the eigenvalue q_0. He found that

$$G(q,q_0,t)=[M\Omega/(2\pi\hbar\mid\sin\Omega t\mid)]^{1/2}$$
$$\times\exp\{iM\Omega[(q^2+q_0^2)\cos\Omega t-2qq_0]/(2\hbar\sin\Omega t)\} \tag{42}$$

develops from a delta function at $t=0$ to a plane wave at $\Omega t=\pi/2$ and back to a delta function at $\Omega t=\pi$, etc. Thus even though the wave packet always returns to its initial state in one period of the oscillator, it has a spread which is a strong function of time. In contrast, however, a wave packet which maintains the same variance while undergoing simple harmonic motion evolves from the ground-state wave function displaced by a distance a:

$$\psi(q,0)=(\alpha^{1/2}/\pi^{1/4})\exp[-\tfrac{1}{2}\alpha^2(q-a)^2]. \tag{43}$$

[35] E. H. Kennard, Z. Physik **44**, 326 (1927).

Then

$$\psi(q,t)=(\alpha^{1/2}/\pi^{1/4})\ \exp\{-\tfrac{1}{2}i\Omega t-\tfrac{1}{2}\alpha^2[(q-a\cos\Omega t)^2+i(aq\sin\Omega t+\tfrac{1}{2}a^2\sin 2\Omega t)]\},\qquad(44)$$

and the probability density is

$$|\psi(q,t)|^2=(\alpha/\pi^{1/2})\ \exp[-\alpha^2(q-a\cos\Omega t)^2].\qquad(45)$$

It may be seen that this packet has the minimum uncertainty product $\Delta q\Delta p=\hbar/2$ allowed by quantum mechanics.

From (17), the average electric field for this wave packet is

$$\langle E\rangle=\sqrt{2}\mathcal{E}\alpha\sin Kz\int_{-\infty}^{\infty}q\,|\psi(q,t)|^2\,dq$$
$$=\sqrt{2}\mathcal{E}\alpha\sin Kz\cos\Omega t.\qquad(46)$$

These states provide the closest quantum-mechanical analog for a free classical single-mode field, and are in fact the coherent states $|\alpha\rangle$ [36,37]:

$$|\psi\rangle=|\alpha\rangle=\sum_n\{[\alpha\exp(-i\Omega t)]^n/(n!)^{1/2}\}\times\exp(-\tfrac{1}{2}|\alpha|^2)\,|n\rangle.\qquad(47)$$

C. Statistical Properties of the Radiation Field

Up to now we have been considering a field which could be represented by a single-state vector $|\psi\rangle$ for which the quantum-mechanical average of an operator Q is

$$\langle Q\rangle=\langle\psi|Q|\psi\rangle.\qquad(48)$$

In general, however, we do not know the exact wave function of our system but rather only the probability P_ψ that our system might have this wave function.[38] The ensemble averaged expression for Q is then

$$\langle\langle Q\rangle\rangle_{\text{ensemble}}=\sum_\psi P_\psi\langle\psi|Q|\psi\rangle,\qquad(49)$$

which may be written as

$$\langle\langle Q\rangle\rangle_{\text{ensemble}}=\mathrm{Tr}(\rho Q),\qquad(50)$$

where

$$\rho=\sum_\psi P_\psi|\psi\rangle\langle\psi|\qquad(51)$$

is the weighted projection operator for the states $|\psi\rangle$. This operator ρ represents our state of knowledge or ignorance about the system. In the n representation this operator becomes a matrix $\rho_{n,n'}$ with an infinite number of rows and columns labeled by the integers 1, 2, 3, $\cdots$.

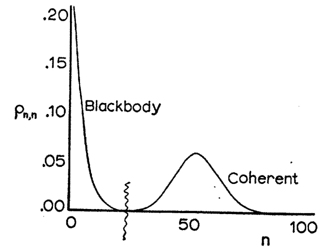

FIG. 2. The photon statistical distribution for single-mode black-body light, Eq. (57), is compared to that for coherent radiation, Eq. (54). The wavy line indicates that two separate curves are shown in the figure.

A few pertinent examples of the density matrix for single-mode light are now given. (1) The field might be in a pure number state

$$\rho_{n,n'}=\delta_{n,n'},\qquad(52)$$

or (2) a pure coherent state

$$\rho_{n,n'}=\langle n|\alpha\rangle\langle\alpha|n'\rangle,\qquad(53)$$

which by (47) is

$$\rho_{n,n'}=\alpha^n\alpha^{*n'}\exp(-|\alpha|^2)/[n!n'!]^{1/2},\qquad(54)$$

or (3) a phase-diffused coherent state

$$\rho_{n,n'}=[(\alpha\alpha^*)^n/n!]\exp(-|\alpha|^2)\delta_{n,n'},\qquad(55)$$

which has no off-diagonal elements. Neither the pure number state nor the phase-diffused coherent ensemble (nor any ρ diagonal in the n representation) has an average electric field, since the ensemble average field involves $\rho_{n,n+1}$:

$$\langle E\rangle\propto\sum_n[\rho_{n,n+1}(n+1)^{1/2}+\text{cc}].\qquad(56)$$

It should be noted that for distributions (3) and (4) the probability for finding n photons $p_n=\rho_{n,n}$ is given by a Poisson distribution characterized by an average n given by $\langle n\rangle=|\alpha|^2$. Another example (4) often met is that of single-mode thermal or black-body radiation of temperature θ:

$$\rho_{n,n'}=\exp(-nh\Omega/k_B\theta)[1-\exp(-h\Omega/k_B\theta)]\delta_{n,n'},\qquad(57)$$

which is diagonal in the n representation and therefore contains no phase information, i.e., has zero-ensemble-average electric field.

It will be noted that the probability of finding n photons in the member of the ensemble under consideration, often called the photon statistics, is radically different for black body and for coherent light. Plots of $\rho_{n,n}$ versus n for coherent radiation, Eq. (54), and for incoherent black-body light, Eq. (57) are given in Fig. 2.

III. MODEL AND ANALYSIS

Having developed the quantum theory of radiation in a form suitable for our purposes, we now turn to

[36] For a discussion of the coherent state formalism see R. J. Glauber, in *Quantum Optics and Electronics; Lectures Delivered at Les Houches During the 1964 Session of the Summer School of Theoretical Physics, University of Grenoble*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965).

[37] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965), give an extensive review of the coherence properties of optical fields. See the bibliography of this paper for references to earlier work.

[38] See for example, U. Fano, Rev. Mod. Phys. **29**, 74 (1957); or D. ter Haar, Rept. Progr. Phys. **24**, 305 (1961).

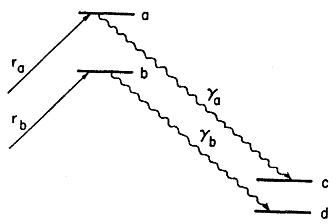

FIG. 3. Atomic-level scheme for atoms. Maser action takes place between levels a and b which are decaying to levels c and d, with decay constants given by γ_a and γ_b, respectively. The corresponding excitations to levels a and b are given by r_a and r_b, respectively.

the fully quantum-mechanical theory of a laser. Both the radiation field and the atomic medium are to be treated according to the laws of quantum mechanics. For simplicity, we consider a gas laser with a single-cavity mode. We neglect the motion of the atoms and spatial variation of the cavity mode. These are non-essential simplifications. The basic idea is the same as in the semiclassical theory. In the earlier work the radiation was described using amplitudes, phases, and frequencies, but now the radiation field has to be characterized in proper quantum-mechanical terminology, i.e., by a density matrix.

To describe laser oscillation, the theory must include a nonlinear active medium and a damping mechanism. To obtain laser pumping action we introduce two-level atoms in their upper state $|a\rangle$ at random times t_0. The more general case of excitation of both the $|a\rangle$ and $|b\rangle$ levels will be dealt with in a later publication. The details of the dissipation mechanism are not very important for the theory of a laser. In the semiclassical theory the damping was represented by Ohmic currents, but it is more convenient for our present purposes to include the dissipation by coupling the electromagnetic field to rapidly decaying, and therefore nonresonant, atoms injected into the cavity in the lower $|\beta\rangle$ of two states $|\alpha\rangle$ and $|\beta\rangle$.

One way of looking at the semiclassical theory is that each atom contributes its mite to the field independently, except insofar as the other atoms have prepared an electromagnetic field with which it interacts. Similarly, in the quantum theory we consider the change in the density matrix for the radiation field due to the injection at time t_0 of a single pumping atom in the upper of the two states $|a\rangle$ and $|b\rangle$ involved in the laser interaction. Working in the n representation, this change is given by

$$\delta\rho_{n,n'}=\rho_{n,n'}(t_0+T)-\rho_{n,n'}(t_0), \qquad (58)$$

where T is a time which is long compared with the atomic lifetime, but short compared to the time characterizing the growth or decay of the laser radiation.

The states $|a\rangle$ and $|b\rangle$ of the atom are assumed to decay as in the Wigner–Weisskopf theory of radiation damping. For the state $|a\rangle$, we introduce a state $|c\rangle$ to which the atom decays with the emission of (non-laser) radiation of type s with a decay constant γ_a. Similarly $|b\rangle$ decays to state $|d\rangle$ with a decay constant γ_b (see Fig. 3).

To obtain $\rho_{n,n'}(t_0+T)$, we must follow the time development of the combined atom-laser field system to time t_0+T and then form the trace of its density matrix over the atomic states

$$\rho_{n,n'}(t_0+T)=\sum_{\alpha}\rho_{\alpha n,\alpha n'}(t_0+T), \qquad (59)$$

where α takes on the values of a, b, c, and d.

Proceeding to calculate $\rho_{n,n'}$ we write the Hamiltonian $\hbar H$, for the interaction of the active atom with the single-mode laser field as

$$H=\nu a^\dagger a+W_a\sigma^\dagger\sigma+W_b\sigma\sigma^\dagger+g(a^\dagger\sigma+a\sigma^\dagger) \qquad (60)$$

$$=H_{\mathrm{rad}}+H_{\mathrm{atom}}+V \qquad (61)$$

$$=H_0+V, \qquad (62)$$

where ν is the laser frequency to be determined from the theory[39] and $\hbar W_a$ and $\hbar W_b$ are the atomic energies; the raising and lowering operators

$$\sigma^\dagger=\begin{pmatrix}0&1\\0&0\end{pmatrix} \quad\text{and}\quad \sigma=\begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad (63)$$

operate on the atomic states

$$|a\rangle=\begin{pmatrix}1\\0\end{pmatrix} \quad\text{and}\quad |b\rangle=\begin{pmatrix}0\\1\end{pmatrix}. \qquad (64)$$

The coupling constant is $g=ex_{ab}\mathcal{E}/(\sqrt{2}\hbar)$ which has the dimensions of a frequency, as $\mathcal{E}$ is the electric field (30).

As shown in Appendix I, the Wigner–Weisskopf approximation for our four-level atom interacting with the radiation field yields the following set of equations for the density matrix of the composite system:

$$\dot{\rho}_{a,n;a,n'}=-i[(H_0+V),\rho]_{a,n;a,n'}-\gamma_a\rho_{a,n;a,n'}, \qquad (65a)$$

$$\dot{\rho}_{b,n+1;b,n'+1}=-i[(H_0+V),\rho]_{b,n+1;b,n'+1}-\gamma_b\rho_{b,n+1;b,n'+1}, \qquad (65b)$$

$$\dot{\rho}_{a,n;b,n'+1}=-i[(H_0+V),\rho]_{a,n;b,n'+1}-\gamma_{ab}\rho_{a,n;b,n'+1}, \qquad (65c)$$

$$\dot{\rho}_{b,n+1;a,n'}=-i[(H_0+V),\rho]_{b,n+1;a,n'}-\gamma_{ab}\rho_{b,n+1;a,n'}, \qquad (65d)$$

$$\dot{\rho}_{c,n;c,n'}=\gamma_a\rho_{a,n;a,n'}, \qquad (65e)$$

$$\dot{\rho}_{d,n+1;d,n'+1}=\gamma_b\rho_{b,n+1;b,n'+1}, \qquad (65f)$$

where $\gamma_{ab}=\frac{1}{2}(\gamma_a+\gamma_b)$. We see that Eqs. (65e) and (65f) may be integrated directly to yield

$$\rho_{c,n;c,n'}(t_0+T)=\gamma_a\int_{t_0}^{t_0+T}dt'\rho(t')_{a,n;a,n'}, \qquad (66a)$$

$$\rho_{d,n+1;d,n'+1}(t_0+T)=\gamma_b\int_{t_0}^{t_0+T}dt'\rho(t')_{b,n+1;b,n'+1}. \qquad (66b)$$

[39] The frequency ν is the frequency of the laser radiation and is determined by the theory. In this paper we are most interested in the case $\omega\neq\nu$; then the laser frequency will be the same as that of the free field. The more general problem of $\omega=\nu$ will be given in a future publication.

Concentrating now on the lasing levels [Eqs. (65a)–(65d)], our equations in expanded form are

$$\dot{\rho}_{a,n;a,n'}=-i[(n-n')\nu-i\gamma_a]\rho_{a,n;a,n'}$$
$$-i[V_{a,n;b,n+1}\rho_{b,n+1;a,n'}-\rho_{a,n;b,n'+1}V_{b,n'+1,a,n'}], \quad (67a)$$
$$\dot{\rho}_{a,n;b,n'+1}=-i[(n-n')\nu+(\omega-\nu)-i\gamma_{ab}]\rho_{a,n;b,n'+1}$$
$$-i[V_{a,n;b,n+1}\rho_{b,n+1;b,n'+1}-\rho_{a,n;a,n'}V_{a,n';b,n'+1}], \quad (67b)$$
$$\dot{\rho}_{b,n+1;a,n'}=-i[(n-n')\nu-(\omega-\nu)-i\gamma_{ab}]\rho_{b,n+1;a,n'}$$
$$-i[V_{b,n+1;a,n}\rho_{a,n;a,n'}-\rho_{b,n+1;b,n'+1}V_{b,n'+1;a,n'}], \quad (67c)$$
$$\dot{\rho}_{b,n+1;b,n'+1}=-i[(n-n')\nu-i\gamma_b]\rho_{b,n+1;b,n'+1}$$
$$-i[V_{b,n+1;a,n}\rho_{a,n;b,n'+1}-\rho_{b,n+1;a,n'}V_{a,n';b,n'+1}]. \quad (67d)$$

The term involving $-i(n-n')\nu$ in these equations will now be transformed away by replacing $\rho_{n,n'}$ by $\rho_{n,n'}\exp[-i(n-n')\nu t]$. It should be kept in mind that subsequently $\rho_{n,n'}$ will be in an interaction picture.

We may write Eqs. (67a)–(67d) as a 2×2 matrix equation

$$\dot{\rho}=-i[C\rho-\rho C'], \quad (68)$$

where

$$\rho=\begin{pmatrix}\rho_{a,n;a,n'} & \rho_{a,n;b,n'+1}\\ \rho_{b,n+1;a,n'} & \rho_{b,n+1;b,n'+1}\end{pmatrix}, \quad (69)$$

$$C=\begin{pmatrix}W_a+n\nu-\frac{1}{2}i\gamma_a & V_{a,n;b,n+1}\\ V_{b,n+1;a,n} & W_b+(n+1)\nu-\frac{1}{2}i\gamma_b\end{pmatrix}, \quad (70)$$

and the matrix C' is obtained from C by replacing n by n' and taking the Hermitian conjugate.

The solution of (68) is clearly

$$\rho(t)=\exp[-iC(t-t_0)]\rho(t_0)\exp[iC'(t-t_0)]$$
$$=\exp[-iC(t-t_0)]\begin{pmatrix}\rho_{n,n'}(t_0) & 0\\ 0 & 0\end{pmatrix}\exp[iC'(t-t_0)]. \quad (71)$$

Next we could solve for the eigenvalues and eigenvectors of C and C' which would facilitate evaluation of

$$\rho_{a,n;a,n'}(t) \quad\text{and}\quad \rho_{b,n;b,n'}(t)$$

and by using (66a) and (66b) could calculate

$$\rho_{c,n;c,n'}(t_0+T) \quad\text{and}\quad \rho_{d,n;d,n'}(t_0+T).$$

Then $\rho_{n,n'}(t_0+T)$ is obtained by contraction with respect to the atomic variables as indicated in Eq. (59):

$$\rho_{n,n'}(t_0+T)=\rho_{a,n;a,n'}(t_0+T)+\rho_{b,n;b,n'}(t_0+T)$$
$$+\rho_{c,n;c,n'}(t_0+T)+\rho_{d,n;d,n'}(t_0+T). \quad (72)$$

Instead of following this approach for obtaining the elements of the density matrix, we adopt another method which has the advantage of side-stepping a considerable portion of the algebraic tedium of the first approach and leads to the same result.

We introduce the notation

$$\int_{t_0}^{t_0+T}\rho_{\beta,\beta'}(t')dt'=\sigma_{\beta,\beta'}(t_0+T),$$
$$\beta=1,2 \quad\text{and}\quad \beta'=1,2,$$
$$\sigma_{1;1}=\sigma_{a,n;a,n'},$$
$$\sigma_{1;2}=\sigma_{a,n;b,n'+1},$$
$$\sigma_{2;1}=\sigma_{b,n+1;a,n'},$$
$$\sigma_{2;2}=\sigma_{b,n+1;b,n'+1}. \quad (73)$$

From Eqs. (66a) and (66b) it is then clear that in the present notation the c- and d-state matrix elements are

$$\rho_{c,n;c,n'}(t_0+T)=\gamma_a\sigma_{1;1}, \quad (74a)$$
$$\rho_{d,n+1;d,n'+1}(t_0+T)=\gamma_b\sigma_{2;2}. \quad (74b)$$

The essence of the simpler approach is the conversion of the differential equations (67) into algebraic equations for σ by integrating both sides from t_0 to t_0+T. We find

$$\rho_{a,n;a,n'}(t_0+T)-\rho_{a,n;a,n'}(t_0)=-\gamma_a\sigma_{11}-i[V_{a,n;b,n+1}\sigma_{21}-\sigma_{12}V_{b,n'+1;a,n'}], \quad (75a)$$
$$\rho_{a,n;b,n'+1}(t_0+T)-\rho_{a,n;b,n'+1}(t_0)=-[i(\omega-\nu)+\gamma_{ab}]\sigma_{12}-i[V_{a,n;b,n+1}\sigma_{22}-\sigma_{11}V_{a,n';b,n'+1}], \quad (75b)$$
$$\rho_{b,n-1;a,n'}(t_0+T)-\rho_{b,n+1;a,n'}(t_0)=-[-i(\omega-\nu)+\gamma_{ab}]\sigma_{21}-i[V_{b,n+1;a,n}\sigma_{11}-\sigma_{22}V_{b,n'+1;a,n'}], \quad (75c)$$
$$\rho_{b,n+1;b,n'+1}(t_0+T)-\rho_{b,n+1;b,n+1}(t_0)=-\gamma_b\sigma_{22}-i[V_{b,n+1;a,n}\sigma_{12}-\sigma_{21}V_{a,n';b,n'+1}]. \quad (75d)$$

As we are interested in times $T\gg1/\gamma_{ab}$, the first term on the left-hand side of each of Eqs. (75a)–(75d) vanishes. Also we note that

$$\rho_{a,n;a,n'}(t_0)=\rho_{n,n'}(t_0), \quad (76)$$

while the other elements of the density matrix with argument t_0 vanish as the initial excitation is to state $|a\rangle$.

In matrix form, Eqs. (75a)–(75d) are now

$$\begin{pmatrix} -i\gamma_a & -V_{b,n'+1;a,n'} & V_{a,n;b,n+1} & 0 \\ -V_{a,n';b,n'+1} & +(\omega-\nu)-i\gamma_{ab} & 0 & V_{a,n;b,n+1} \\ V_{b,n+1;a,n} & 0 & -(\omega-\nu)-i\gamma_{ab} & -V_{b,n'+1;a,n'} \\ 0 & V_{b,n+1;a,n} & -V_{a,n';n'+1} & -i\gamma_b \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} -i\rho_{n,n'}(t_0) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{77}$$

The problem has thus been reduced to solving four simultaneous algebraic equations with four unknowns, which is easily accomplished by matrix techniques. Solving for σ_{11} and σ_{22} we find, with relatively little effort,

$$\gamma_a\sigma_{1;1}=\rho_{nn'}(t_0)-[(n+1)\mathcal{R}_{n,n'}+(n'+1)\mathcal{R}_{n',n}{}^*]\rho_{n,n'}(t_0), \tag{78a}$$

$$\gamma_b\sigma_{2;2}=[\mathcal{R}_{n,n'}+\mathcal{R}_{n',n}{}^*][(n+1)(n'+1)]^{1/2}\rho_{n,n'}(t_0), \tag{78b}$$

where

$$\mathcal{R}_{n,n'}=g^2\frac{\gamma_b(\gamma_{ab}+i\Delta)+g^2(n-n')}{\gamma_a\gamma_b(\gamma_{ab}{}^2+\Delta^2)+2\gamma_{ab}{}^2g^2(n+1+n'+1)+g^2(n'-n)[g^2(n'-n)+i\Delta(\gamma_a-\gamma_b)]} \tag{79}$$

with $\Delta=\omega-\nu$.

We are now in a position to calculate $\rho_{n,n'}(t_0+T)$ as given by Eq. (72). For reasons already mentioned, the first two terms of (72) vanish. Using (74a), and (74b) with $n\to n-1$ and $n'\to n'-1$, we have

$$\rho_{n,n'}(t_0+T)=\gamma_a\sigma_{1;1}+\gamma_b\sigma_{2;2}$$
$$(\text{with } n\to n-1 \text{ and } n'\to n'-1). \tag{80}$$

Thus we have all the ingredients needed to obtain the change in the radiation-field-density matrix $\delta\rho_{n,n'}$ as given by Eq. (58). From Eqs. (78a), (78b) and (80) we find

$$\begin{aligned}\delta\rho_{n,n'}&=\rho_{n,n'}(t_0+T)-\rho_{n,n'}(t_0)\\ &=-[(n+1)\mathcal{R}_{n,n'}+(n'+1)\mathcal{R}_{n',n}{}^*]\rho_{n,n'}\\ &\quad+[\mathcal{R}_{n-1,n'-1}+\mathcal{R}_{n'-1,n-1}{}^*](nn')^{1/2}\rho_{n-1,n'-1}.\end{aligned} \tag{81}$$

Just as in the classical theory where we have taken the phase and amplitude of the field to be slowly varying, in the quantum theory the density matrix for the field will not change much due to one atom. Hence we note that for the time interval $t_0<t<t_0+T$, we have $\rho_{n,n'}(t_0)\approx\rho_{n,n'}(t)$. To obtain a macroscopic change in the density matrix, we now multiply (81) by the number of atoms entering the cavity in a time Δt which is long compared to an atomic lifetime but short compared to times characterizing the growth or decay of the radiation field. The number of atoms injected in the upper level in a time Δt is $N_a=r_a\Delta t$, i.e., the rate r_a of injection multiplied by the time Δt. Then the macroscopic change in $\rho_{n,n'}$ due to many atoms acting on the field is

$$\begin{aligned}\Delta\rho_{n,n'}&=r_a\delta\rho_{n,n'}\\ &=-\Delta t[(n+1)R_{n,n'}+(n'+1)R_{n',n}{}^*]\rho_{n,n'}(t)+\Delta t[R_{n-1,n'-1}+R_{n'-1,n-1}{}^*](nn')^{1/2}\rho_{n-1,n'-1}(t),\end{aligned} \tag{82}$$

where $R_{n,n'}=r_a\mathcal{R}_{n,n'}$. The coarse-grained time derivative due to many atoms interacting with the field is then

$$\begin{aligned}[d\rho_{n,n'}/dt]_{(\text{stimulated and spontaneouse mission})}&=-[(n+1)R_{n,n'}+(n'+1)R_{n',n}{}^*]\rho_{n,n'}(t)\\ &\quad+[R_{n-1,n'-1}+R_{n'-1,n-1}{}^*](nn')^{1/2}\rho_{n-1,n'-1}(t).\end{aligned} \tag{83}$$

A similar analysis follows for the dissipative interaction, but as mentioned earlier, the details are of secondary interest and we relegate the calculation to Appendix II. We find (to second order in the coupling) from Eq. (II.5)

$$\begin{aligned}[d\rho_{n,n'}/dt]_{\text{dissipation}}&=-\tfrac{1}{2}C(n+n')\rho_{n,n'}\\ &\quad+C[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1},\end{aligned} \tag{84}$$

where the quantity $C=\nu/Q$ is the cavity band width.

Finally, we write the complete equations of motion for the radiation-density matrix (in the interaction picture) as

$$\begin{aligned}d\rho_{n,n'}/dt=&-[(n+1)R_{n,n'}+(n'+1)R_{n',n}{}^*]\rho_{n,n'}\\ &+[R_{n-1,n'-1}+R_{n'-1,n-1}{}^*](nn')^{1/2}\rho_{n-1,n'-1}\\ &-\tfrac{1}{2}C(n+n')\rho_{n,n'}+C[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1}.\end{aligned} \tag{85}$$

Equations (85) are the basic results of this section and provide the quantum equivalent of the classical amplitude and phase equations, with C_n and S_n determined by a self-consistent-field analysis.

IV. DISCUSSION OF EQUATIONS OF MOTION AND PHOTON STATISTICS

It will be noticed that the equations of motion (85) couple only elements of the density matrix having equal degree of off-diagonality $n-n'$, i.e., the coupling is along lines parallel to the main diagonal. Taking advantage of this decoupling, we now investigate the diagonal equations $n=n'$ obtained from (85). For simplicity, in the remainder of this paper we will consider the laser to be tuned to atomic resonance. Detuning and other complications will be discussed in a later paper of this series. These diagonal equations are

$$\dot{\rho}_{n,n}=-A(n+1)[1+(n+1)(B/A)]^{-1}\rho_{n,n}+An[1+n(B/A)]^{-1}\rho_{n-1,n-1}-Cn\rho_{n,n}+C(n+1)\rho_{n+1,n+1}, \quad (86)$$

where

$$A=2r_a(g^2/\gamma_a\gamma_{ab}), \quad (87)$$

$$B=8r_a(g^2/\gamma_a\gamma_{ab})(g^2/\gamma_a\gamma_b), \quad (88)$$

$$C=\nu/Q. \quad (89)$$

Equations (86) describe the flow of probability for finding n photons in the laser cavity. The separate terms representing the time rates of change of probability have been grouped to make the physical interpretation obvious as depicted in Fig. 4.

These equations for $\rho_{n,n}(t)$ have transient solutions which would describe, for example, the buildup from vacuum to a steady state. We will limit the discussion here to the steady-state solution. By inspection of Fig. 4 it is clear that detailed balance implies that these second-order difference equations reduce to the two equivalent systems of first-order difference equations

$$An[1+(B/A)n]^{-1}\rho_{n-1,n-1}-Cn\rho_{n,n}=0, \quad (90)$$

$$A(n+1)[1+(B/A)(n+1)]^{-1}\rho_{n,n}-C(n-1)\rho_{n+1,n+1}=0. \quad (91)$$

The solution of these equations is clearly

$$\rho_{n,n}=\mathfrak{N}\prod_{k=0}^{n}(A/C)[1+(B/A)k]^{-1}, \quad (92)$$

where $\mathfrak{N}$ is a normalization constant.

Let us consider this distribution in three regions of laser operation:

$A>C$ (*above threshold*). The quantity $\rho_{n,n}$ is the product of $n+1$ factors of the form $(A/C)[1+(B/A)k]^{-1}$. For $k<(A/C)(A-C)B^{-1}=n_p$, these factors are each greater than unity, while for $k>n_p$ the factors $(A/C)[1+(B/A)k]^{-1}$ are less than unity; hence $\rho_{n,n}$ increases for n up to n_p and goes monotonically to zero for $n>n_p$. Thus the distribution is peaked at n_p.

$A=C$ (*threshold*). The distribution $\rho_{n,n}$ has a maximum at $n=0$ and decreases in a roughly Gaussian fashion as n increases.

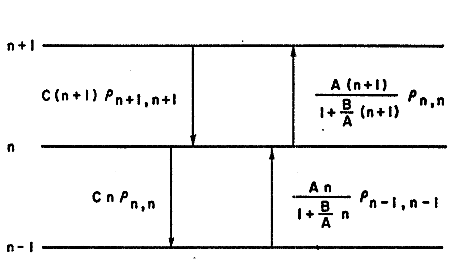

FIG. 4. Flow of probability for finding n photons in the laser cavity due to stimulated emission and damping.

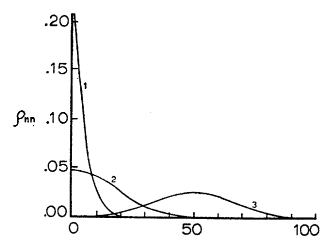

FIG. 5. The laser distribution, Eq. (96), illustrating the three operating regions: (1) 20% below threshold, (2) threshold, and (3) 20% above threshold. Using Eq. (99) the nonlinear parameter B has been chosen to give $\langle n\rangle=50$ at 20% above threshold. The laser distribution (3) should be compared to that for coherent light in Fig. 3.

$A<C$ (*below threshold*). Now the distribution falls more rapidly to zero. In this region the nonlinear terms may be ignored and we write

$$\rho_{n,n}=[1-(A/C)](A/C)^n. \quad (93)$$

Hence, below threshold the steady-state solution is essentially that of a black-body cavity

$$\rho_{n,n}=[1-\exp(-\hbar\nu/k_B\theta)]\exp(-n\hbar\nu/k_B\theta), \quad (94)$$

where the effective temperature θ is defined by

$$\exp(-\hbar\nu/k_B\theta)=A/C. \quad (95)$$

The photon distribution in these three regions is displayed in Fig. 5. The steady-state distribution (92) with $\langle n\rangle=10^6$ is compared with a coherent distribution of the same mean value in Fig. 6.

We may write Eq. (92) in a more convenient form as

$$\rho_{nn}=Z^{-1}(A^2/BC)^{n+(A/B)}/[n+(A/B)]!, \quad (96)$$

where the normalization constant Z^{-1} may be expressed in terms of confluent hypergeometric functions

$$Z=\sum_{n=0}^{\infty}\frac{(A^2/BC)^{n+(A/B)}}{[n+(A/B)]!}=\left[\frac{(A^2/BC)^{A/B}}{(A/B)!}\right]\left[{}_1F_1\left(1;\frac{A}{B}+1;\frac{A^2}{BC}\right)\right]. \quad (97)$$

Calculating the average value of n, we find

$$\begin{aligned}\langle n\rangle&=Z^{-1}\sum_{n=0}^{\infty}n\frac{(A^2/BC)^{n+(A/B)}}{[n+(A/B)]!}\\&=Z^{-1}\sum_{n=1}^{\infty}\left(n+\frac{A}{B}-\frac{A}{B}\right)\frac{(A^2/BC)^{n+A/B}}{(n+A/B)!}\\&=Z^{-1}\sum_{n=1}^{\infty}\left[\frac{(A^2/BC)^{n+(A/B)-1}}{(n+A/B-1)!}\right]\frac{A^2}{BC}-\frac{A}{B}(1-\rho_{0,0})\\&=\left[\sum_{n=0}^{\infty}\rho_{n,n}\right]\frac{A^2}{BC}-\frac{A}{B}(1-\rho_{0,0})\\&=(A/C)(A-C)B^{-1}+(A/B)\rho_{0,0}.\end{aligned} \quad (98)$$

For a laser appreciably above threshold, the $(A/B)\rho_{0,0}$ term in (98) is clearly insignificant because $\rho_{0,0}\ll 1$, and we have

$$\langle n\rangle=(A/C)(A-C)B^{-1}. \quad (99)$$

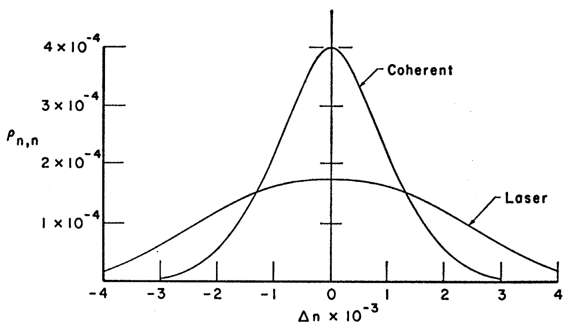

FIG. 6. This figure compares the photon statistics for coherent and laser radiation. The laser is here taken to be 20% above threshold, with the parameter B chosen to give $\langle n\rangle = 10^6$.

A similar approximation for the variance of the distribution yields

$$\sigma^2 = [A/(A-C)]\langle n\rangle. \tag{100}$$

For a gas laser not too far above threshold, Eqs. (86) may be adequately approximated by retaining only the lowest-order terms in B/A and one finds

$$\dot{\rho}_{n,n} = -[A-B(n+1)](n+1)\rho_{n,n} + [A-Bn]n\rho_{n-1,n-1} - Cn\rho_{n,n} + C(n+1)\rho_{n+1,n+1}. \tag{101}$$

The steady-state solution for (101) is

$$\rho_{n,n} = \mathfrak{N}' \prod_{k=0}^{n} \frac{A-Bk}{C}, \tag{102}$$

where $\mathfrak{N}'$ is the normalization constant. This distribution has a peak at

$$n_p = (A-C)/B. \tag{103}$$

For a sufficiently peaked distribution the average value $\langle n\rangle$ obtained from (103) is

$$\langle n\rangle \approx n_p = \frac{(\text{linear pumping}) - (\text{damping})}{(\text{nonlinear parameter})}.$$

We see that the average energy contained in the laser $\langle n\rangle\hbar\nu$ corresponding to this $\langle n\rangle$ has a direct counterpart in Eq. (16a) which expresses the energy $\frac{1}{4}\epsilon_0 E^2 AL$ of the semiclassical theory.

It will be noted that a consequence of the expansion in (B/A) is that for very large values of n, i.e., $n > A/B$, the distribution $\rho_{n,n}$ goes negative; however, this is well beyond the range of interest, $n = \langle n\rangle \pm O(\langle n\rangle^{1/2})$, and should cause no alarm. Furthermore the difficulty can be avoided by merely letting A/B have an integral value, as this will insure $\rho_{n,n} = 0$ for $n > A/B$.

To investigate the electric field of the laser, we must turn to the off-diagonal elements of the density matrix.

V. OFF-DIAGONAL ELEMENTS, CORRELATION TIMES AND SPECTRAL PROFILE

Equations (85) have an infinite number of exponential decaying solutions corresponding to different decay eigenvalues $\mu_s^{(k)}$. These are of the form

$$\rho_{n,n+k}{}^{s} = \varphi_s(n, k)\exp(-\mu_s^{(k)}t). \tag{104}$$

For the diagonal elements, $k=0$, the lowest eigenvalue $\mu_0^{(0)} = 0$ and the corresponding eigenfunction is the steady-state solution (96). For the off-diagonal elements all of the eigenvalues $\mu_s^{(k)}$ are positive. Consequently, the only steady-state solution is

$$\rho_{n,n'} = 0, \qquad n \neq n'. \tag{105}$$

It is planned to give a full discussion of these transient solutions (104) in a later paper, but here we will confine our attention to the slowest decay modes for $n \neq n'$. Consequently, for many purposes we may write

$$\rho_{n,n+k} = \varphi_0(n, k)\exp(-\mu_0^{(k)}t). \tag{106}$$

In Appendix III it is shown that the desired eigenfunction for a laser sufficiently above threshold is

$$\varphi_0(n, k) = \left\{\prod_{l=0}^{n}\left[\frac{A-Bl}{C}\right]\prod_{m=0}^{n+k}\left[\frac{A-Bm}{C}\right]\right\}^{1/2} \tag{107}$$

and, to a good approximation, the corresponding eigenvalue is

$$\mu_0^{(k)} = \tfrac{1}{2}k^2 D, \tag{108}$$

where

$$D = \tfrac{1}{2}(\nu/Q)\langle n\rangle^{-1}. \tag{109}$$

From (106) and (108) we may then write

$$\rho_{n,n+k}(t) = \rho_{n,n+k}(0)\exp(-\tfrac{1}{2}k^2 Dt). \tag{110}$$

The expectation value of the electric field for this density matrix is given by

$$\begin{aligned} E(z,t) &= \mathcal{E}\sin(s\pi z/L)\sum_n (\rho_{n,n+1}(t)(n+1)^{1/2} + \text{c.c.}) \\ &= \mathcal{E}\sin(s\pi z/L)\sum_n (\rho_{n,n+1}(0)(n+1)^{1/2}e^{-i\nu t} + \text{c.c.})e^{-Dt/2} \\ &= E_0\exp[-\tfrac{1}{2}Dt]\cos\nu t. \end{aligned} \tag{111}$$

To obtain the line shape we take the Fourier transform (in the rotating wave approximation) of the average E field.

$$E(\omega) = \int_0^\infty e^{-i\omega t}E_0\exp[-\tfrac{1}{2}Dt]\cos\nu t\,dt \tag{112}$$

$$= E_0[i(\omega-\nu) + \tfrac{1}{2}D]^{-1}, \tag{113}$$

and the spectral profile is

$$|E(\omega)|^2 = E_0^2[(\omega-\nu)^2 + (\tfrac{1}{2}D)^2]^{-1}. \tag{114}$$

Thus the spectral profile for the laser oscillator is Lorentzian with a width

$$D = \tfrac{1}{4}(\nu/Q)\langle n\rangle^{1/2}, \tag{115}$$

which is the full-width at half-height, in circular frequency units; see Fig. 7. The physical interpretation of this linewidth and the associated decaying electric field will be found in Sec. VI.

A comparison of the present expression for the spectral width D and that derived previously by modifying the semiclassical theory to include noise[17] indicates that we now have twice as much width. From the structure of the calculation, it is apparent that the doubling comes because the present treatment includes the noise due to spontaneous emission of the active atoms as well as thermal and zero-point fluctuations in the cavity walls. The present linewidth is in agreement with the independent results of Korenman[40] and Lax.[41]

VI. DISCUSSION

A. Nature of the Problem

Theoretical physics is most fully developed for treatment of the behavior of isolated conservative dynamical systems. The addition of a given conservative external force field does not present much difficulty, at least in principle. As one attempts to make the discussion of a problem more realistic, the system of interest may be allowed to interact with a thermal reservoir, thereby developing a thermodynamic or statistical mechanical approach. After passage of a sufficiently long time, the system of interest settles down into a state of equilibrium. In cooperative phenomena, the equilibrium state may be one with a sigh degree of long-range spatial order. Such theories predict that statistical fluctuations should occur about the thermodynamic equilibrium state.

An oscillating laser, on the other hand, is clearly not in a state of random fluctuations about thermodynamic equilibrium. It represents an "open" system with a highly organized temporal behavior. The system of interest is in contact with a steady but nonthermal reservoir capable of supplying energy at at a low frequency which is somehow converted into a nearly monochromatic oscillatory behavior at an optical frequency. Such problems push somewhat beyond the range of present-day theoretical physics, and one can make progress only by exploiting some special simplifying feature of the problem. Only much later can one expect to succeed with problems in which the openness permits exchange of both energy and matter, as would be necessary for a basic discussion of a biophysical problem.

B. Model

There are two special features simplifying our model. The first is that the electromagnetic field of a high-Q optical resonator is dynamically equivalent to a system with a single degree of freedom, in this case, a simple harmonic oscillator. The second feature is that under circumstances of interest in laser physics, the density matrix describing the radiation does not change very much during the lifetime of one atom.

Our model for the pumping mechanism is quite realistic for a gaseous optical maser. The excitation of atoms to the laser states $|a\rangle$ or $|b\rangle$ is, as far as the system of interest is concerned, essentially an act of

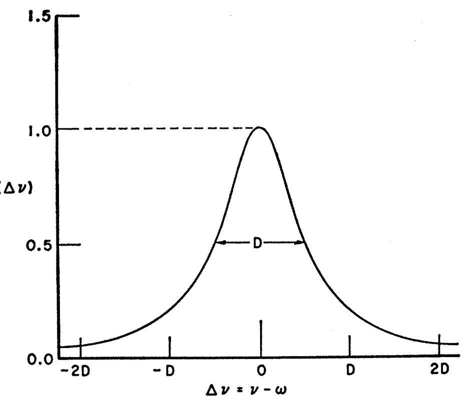

FIG. 7. Spectral profile for a laser oscillator in units of the full-width at half-height D, all frequencies are measured in circular-frequency units.

creation as assumed in the model. In order to simplify the following discussion the case of a excitation will be considered. Assuming that the density matrix describing the radiation field just before the atom is injected is $\rho_{n,n'}(t_0)$, we have calculated the time development of the density matrix for the combined system of atom and field. After several atomic lifetimes, the atom is surely in one of the lower states, c or d, and the only nonvanishing elements of the density matrix are of the form $\rho_{d,n+1;d,n'+1}$ or $\rho_{c,n;c,n'}$. If one then asks for the statistical density matrix describing the radiation field alone, irrespective of whether the atom has ended up in state $|d\rangle$ or $|c\rangle$, i.e., whether the atom did or did not emit a net laser quantum before it decayed, the result is

$$\rho_{n,n'}(t_0+T)=\rho_{c,n;c,n'}(t_0+T)+\rho_{d,n;d,n'}(t_0+T).$$

This process of contraction is an essential feature of the model. A consequence is that even if the radiation field were initially described by a "pure" case density matrix $\rho_{n,n'}(t_0)$, after the injection and decay of one reservoir atom, the density matrix $\rho_{n,n'}(t_0+T)$ would in general be "mixed." One could, of course, in principle, learn more about the combined system by observing whether the atom ended in state $|c\rangle$ or in state $|d\rangle$, but our object is to make a theory describing the system of interest, which is the laser and not the pumping reservoir.

In our theory of a laser, it is not necessary to postulate any noise sources either of a quantum or a classical nature. The noise is automatically produced as a consequence of the contraction process which is basic to the physical problem and follows from the principles of quantum mechanics applied to an open system. There are, however, fluctuations of a shot-effect nature which arise from the injection of pumping and damping atoms. These will be treated in a subsequent paper.

After the change $\delta\rho_{n,n'}$ of the density matrix is calculated for one atom, we pass over to a coarse-grained time derivative due to many atoms. It is thereby implied that $\rho_{n,n'}(t)$ does not change much during the

[40] V. Korenman, Ref. 11, p. 748.
[41] M. Lax, Ref. 11, p. 735.

life of any one atom. This may or may not be the case for any given laser. At a steady state, the changes $\delta\rho_{n,n'}$ represent fluctuations which would typically be small fractions of $\rho_{n,n'}$. Our assumption might be less valid in a transient problem. Thus, if initially we had the vacuum radiation field ($\rho_{0,0}=1$, $\rho_{n,n}=0$ for $n\neq 0$), the first atom would change $\rho_{0,0}$ and $\rho_{1,1}$ by small but finite amounts.

The change in $\rho_{n,n'}$ due to other atoms during the life of the atom considered in the calculation of $\rho_{n,n'}$ would involve an approximation in the derivation of Sec. III. A corresponding simplification was made in the semiclassical theory. Thus, in Eq. (37) of Ref. 1, the amplitude $E(t')$ in an integral

$$\int_{-\infty}^{t} dt' E(t') \cdots$$

is replaced by $E(t)$. This assumes that $E(t)$ is slowly varying during an effective atomic lifetime. Except for the earliest stages of buildup from the vacuum state, the validity of our treatment depends on the smallness of the quantity $\{A-(\nu/Q)\}/\gamma_{ab}$, which is 0.01 for the numerical values $A=1.1$ MHz, ($\nu/Q=1.0$ MHz and $\gamma_{ab}=10$ MHz).

C. Approach to Thermal Equilibrium

If we neglect all nonlinear terms involving saturation of the atomic transitions [set $B=0$ in Eq. (91)], our model describes a harmonic-oscillator system of interest in contact with two reservoirs. One of these contains a large number of pumping atoms in the upper laser state $|a\rangle$, and the other contains many damping atoms in the lower state $|\beta\rangle$. One can assign temperatures to each reservoir in the conventional manner. The first reservoir is at $T=-0°$K (very hot) and the second is at $T=0°$K (very cold). Both pumping and damping atoms are separately injected in the optical cavity, and it is assumed that the density matrix for the radiation field is not changed very much by any one atom. In a steady state, the density matrix is

$$\rho_{n,n}\propto (A/C)^n.$$

If $A<C=\nu/Q$, this can be normalized and is a thermodynamic distribution

$$\rho_{n,n}=(A/C)^n[1-(A/C)],$$

corresponding to a temperature θ given by

$$\exp[-(\hbar\nu/k_B\theta)]=A/C\leq 1.$$

The situation is somewhat different from that usually considered, in that the effects of two reservoirs, one hot and one cold, are combined in order to keep the system of interest at an intermediate temperature θ which depends on the strengths A and C of the coupling between the radiation oscillator and the two reservoirs. For the system of interest, the steady-state thermodynamic equilibrium at temperature θ is the same whether one conventional reservoir is involved or the two reservoirs of our model. If the system of interest is not initially in thermal equilibrium, its approach to this state can be determined by solving the equations of motion (91).

This calculation adds another example to a short list of solvable problems where approach to thermal equilibrium is considered in a basic manner. It is similar to the Rayleigh[42] problem of a massive particle sent into a gas of light atoms. Even closer to the laser problem is the generalization of Uhlenbeck and Chang,[43] where a forced simple harmonic oscillator is brought to steady state through collisions with such a gas.

D. The Case of Nonthermal Equilibrium A>C

It is perfectly possible for a two-level system to have a negative temperature if $\rho_{a,a}>\rho_{b,b}$, but it is meaningless for a system, such as a harmonic oscillator, whose energy spectrum has no upper limit to have a negative temperature. Formally, one may try to make θ negative in Eq. (91) with $B=0$ by setting $A>C$, but $\rho_{n,n}$ would then be a steeply increasing function of n, and the photon statistical distribution could not be normalized. To avoid this difficulty, it is necessary to retain the nonlinear B terms in the steady-state solution for $\rho_{n,n}$. Above threshold, the laser photon distribution does rise with increasing n to a peak at $n=n_p$ beyond which saturation effects play an essential role, and bring the distribution down again.

E. Measurement

Quantum electrodynamics is based on a generalization of nonrelativistic quantum mechanics. Despite the analysis given by Bohr and Rosenfeld, it is probably safe to say that the theory of measurement is even less well developed for quantum electrodynamics than for nonrelativistic quantum mechanics.[44] Some of the difficulty, no doubt, arises from the infinite number of degrees of freedom of quantum electrodynamics. In the particular case of a single high-Q cavity mode, however, it seems possible to regard the analogy between a radiation oscillator and a mechanical oscillator as so close that the measurement problems become equivalent. The following discussion is based on this assumption.

The most that can possibly be known about the radiation oscillator at $t=0$ is its wave function, say $\psi(E,0)$ in the electric field E "coordinate" representation. We assume that this state has been "prepared" somehow. Under the guidance of a definite Hamiltonian, this wave function will evolve into $\psi(E,t)$ at time t. Any Hermitian operator $F(E,-i\hbar\partial/\partial E)$ can, or so the theory contends, be "measured." Each measurement gives as a result one of the eigenvalues F_n of

[42] Lord Rayleigh, Phil. Mag. **32**, 424 (1891).

[43] G. E. Uhlenbeck and C. S. W. Chang, in *Proceedings of the International Synposiun on Transport Processes in Statistical Mechanics*, edited by I. Prigogine (Interscience Publishers, Inc., New York, 1958), pp. 161–168.

the operator F. The probability of finding a particular value F_n when a series of measurements is made on an ensemble of similarly prepared systems is

$$W_n = \left| \int_{-\infty}^{\infty} dE\, \varphi_n(E)^* \psi(E, t) \right|^2,$$

where $\varphi_n(E)$ is the eigenfunction belonging to the eigenvalue F_n of the operator F. The measurements under discussion here are the best permitted. If carried out well, a measurement so disturbs the system of interest that it is pointless to even think of any subsequent measurement of any other operator. The pure case will become a hopeless mixture, even if the system is not physically destroyed. In most physical research, one is not concerned with measurement in this extreme form. Certain scattering experiments are sometimes called measurements, but they do not represent measurement of a Hermitian operator in the strict sense, so we would prefer to call them observations or "bad" measurements. Sometimes, especially when a nearly classical system is under study, one attempts to follow the time development between $t=0$ and $t=t$ by making a series of observations. In our opinion, there is currently no satisfactory theory[44] of "bad" measurements. We do recognize the possibility of "watching" the bob of a pendulum clock swing back and forth. In a similar manner, at least in principle, the temporal oscillations of the intense and highly classical electric field in a laser could be followed by recording on a moving film the deflection of a stream of high-velocity electrons sent across a narrow laser beam.

We have noted in Eq. (109) that the ensemble average of $E(t)$ is a damped oscillating function of time. This damping comes from phase diffusion of the fields for an ensemble of lasers which represents various possible histories of any one laser. An electron-beam probe of any one continuous wave laser would, of course, not show such a dampling, but only a very slight amount of phase irregularity. The average of many similar film records would naturally show the damping phenomenon.

F. Spectrum

In Eq. (112) we have calculated the spectrum associated with the damped oscillating electric field (109) and have found a Lorentzian of full-width at half-maximum just equal to the phase-diffusion constant D. If one had a laser described by a purely diagonal density matrix $\rho_{n,n'}$, the average $\langle E(t)\rangle$ would be zero, and it might seem that a spectrum could not be defined. It is clear that there is no real difficulty here. Any reasonable operational procedure for determining a spectrum would give the desired result. One could, for example, make a Fourier analysis of a very long stretch of the film record mentioned above. The phase information available on the early part of the tracing would, in effect, represent preparation of an ensemble with a nonvanishing off-diagonal density matrix.

[44] W. E. Lamb, Jr. and Y. Aharanov (to be published).

In a subsequent paper, we will work out in detail the theory of a model spectrum analyzer coupled to the laser which does not involve a "bad" measurement of $E(t)$.

G. Photon Statistics

In principle, according to the assumed quantum theory of measurement, one could measure the total amount of energy in the single-mode optical cavity since this is represented by the Hermitian Hamiltonian operator. The result of such a measurement would be an integer multiple n of $\hbar\nu$, apart from the zero-point energy $\frac{1}{2}\hbar\nu$. Each time the measurement was repeated on a similarly prepared system, the n value could change. The statistical distribution of n values after many measurements would be given by the diagonal elements $\rho_{n,n}$ of the density matrix.

In practice, it would not be easy to determine $\rho_{n,n}$ in this manner. As a partial substitute one might count the number of photoelectrons emitted in a certain time interval. In the usual observations, the photoelectron counting is done with the detector located outside the laser cavity and the relationship of the results to $\rho_{n,n}$ is further complicated by diffraction of the radiation escaping from the laser cavity. It would not be very practical, but simpler in principle, to place the photoelectric surface inside the laser cavity. Even here, the photoelectron counting statistics would give a somewhat blurred image of the photon statistical distribution $\rho_{n,n}$. A theory of this process has been given by the authors,[45] and a fuller account is in preparation. The problem of photoelectron counting statistics has also been discussed elsewhere.[46]

H. Absence of Coupling between Elements of the Density Matrix having Different Degrees of "Off Diagonality"

As seen from Eq. (90) the differential equations for $\rho_{n,n'}(t)$ separate into systems of equations connecting elements $\rho_{n,n+k}$ of equal "off-diagonality" $k=n'-n$. For example, if ρ is diagonal initially, it will remain so forever. Similarly, off-diagonal elements like $\rho_{n,n+1}$ evolve completely independently of the diagonal elements $\rho_{n,n}$ and vice versa. This separability is a necessary corollary of the fact that ρ can represent our state of knowledge of any ensemble of lasers. Some restrictions on possible initial values of $\rho_{n,n}$ are imposed by general properties of density matrices. Among these are

$$\sum_n \rho_{n,n}=1 \qquad 0\le \rho_{n,n}\le 1, \qquad \text{all } n$$

$$0\le \text{all eigenvalues of } \|\rho_{n,n'}\| \le 1.$$

[45] Presented at the Second Rochester Conference on the Quantum Theory of Optical Coherence, June, 1966 (unpublished).

[46] In this context, see especially: L. Mandel, in *Progress in Optics*, edited by E. Wolf (North-Holland, Publishing Company, Amsterdam, 1963), Vol. II; P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964); R. J. Glauber, Ref. 36.

Since these properties must be satisfied initially, they will continue to be satisfied as the ensemble evolves in time.

I. Symmetry Breaking

In the statistical mechanics of a magnetic substance, the average magnetization at temperature θ is given by an expression like

$$\langle M_z\rangle = \mathrm{Tr}[M_z\rho(\theta)],$$

where $\rho(\theta)$ is the density matrix at equilibrium for the magnetic system.

In the absence of an external magnetic field, the above magnetization is zero on grounds of inversion symmetry. This would seem to rule out the possibility of permanent magnets with a nonzero value of $\langle M_z\rangle$. The resolution of this paradox is well known. The spontaneously magnetized sample is not really in thermodynamic equilibrium, and has to be described by a nonthermodynamic ensemble. Left in contact with a thermal reservoir for a sufficiently long time, the average magnetism of the ensemble would decay to zero, although with a very slow decay rate.

Similar considerations apply to the ensemble average for the quantum laser. General symmetry arguments would lead one falsely to the conclusion that $\langle E(t)\rangle$ should always be zero, but a properly biased ensemble could easily have $\langle E(t)\rangle$ nonzero. The decay time $1/D$ for a typical gas laser is of the order 10^3 sec, which is enormous compared to the period of oscillation 10^{-14} sec.

VII. SUMMARY

This paper develops the quantum theory of a laser oscillator. The radiation field in the cavity resonator is described by a density matrix $\rho_{n,n'}(t)$ in the n representation. A system of differential-difference equations (85) determines the time development of $\rho_{n,n'}(t)$ due to the combined effects of pumping and damping. These are the quantum analog of the amplitude and phase equations (4) in the semiclassical theory, with $S_n(t)$ and $C_n(t)$ determined by a self-consistent field approximation. A steady-state solution for $\rho_{n,n}(\infty)$ is given by Eq. (96), while the off-diagonal elements $\rho_{n,n'}(\infty)=0$ for $n\neq n'$.

In the case of a laser oscillator operating well above threshold, the steady-state photon probability distribution $\rho_{n,n}(\infty)$ is a sharply peaked distribution somewhat broader than a corresponding Poisson distribution. It should be emphasized that the peaked nature of the photon statistical distribution, which is a manifestation of laser coherence, is a result of the nonlinear aspects of the problem. In this case, the off-diagonal elements $\rho_{n,n'}$ $n\neq n'$ decay to their steady state in approximately exponential manner $\exp(-\frac{1}{2}k^2Dt)$ where the decay rate D is given by Eq. (109) and k is the degree of offdiagonality.

The temporal development of the photon statistical distribution $\rho_{n,n}$ has not been discussed in this paper, but has been analyzed by a numerical calculation and the results have been presented in the form of a moving picture.[47] A fuller account of the temporal behavior of $\rho_{n,n'}(t)$ predicted from Eqs. (85) will be presented in a future publication.

APPENDIX I: WIGNER-WEISSKOPF THEORY FOR A FOUR-LEVEL INTERACTING WITH THE LASER FIELD

In the analysis of Sec. III we represented the effect of certain nonlaser modes of the field on our atoms by introducing the radiative damping coefficients γ_a and γ_b. In this appendix we will derive the working relations Eqs. (65) in the Wigner–Weisskopf approximation. We are really interested in the two atomic levels a and b, but in order to consider their radiative decay it is necessary to introduce two more levels c and d. State $|a\rangle$ may decay to $|c\rangle$ with the emission of radiation of frequency ν_s, while $|b\rangle$ goes to $|d\rangle$ with the emission of a photon of frequency ν_σ, as shown in Fig. 8. The concept of an unobserved coordinate or reservoir is nicely illustrated by this example.[48] That the decay radiation is unobserved is clear when we recall that we are looking for $\rho_{nn'}(t)$, which is the trace over all indices except those referring to the laser radiation, i.e.,

$$\rho_{n,n'}(t)=\mathrm{Tr}_\alpha\ \mathrm{Tr}_{\{s,\sigma\}}\rho_{\alpha,\{s,\sigma\},n;\alpha,\{s,\sigma\},n'},\tag{I.1}$$

where $\alpha=a,\ b,\ c,\ d$ is the atomic index and $\{s,\ \sigma\}$ denotes the decay radiation.

Let us consider the interaction of a four-level atom with the laser mode of frequency ν and the continuum of decay modes of frequencies ν_s and ν_σ. The Hamiltonian for this system is

$$\begin{aligned}H=&\nu a^\dagger a+\sum_s\nu_s a_s{}^\dagger a_s+\sum_\sigma\nu_\sigma a_\sigma{}^\dagger a_\sigma+\sum_{\alpha=a,b,c,d}\epsilon_\alpha A_\alpha{}^\dagger A_\alpha\\&+g[a^\dagger A_b{}^\dagger A_a+aA_a{}^\dagger A_b]\\&+\sum_s g_s[a_s{}^\dagger A_c{}^\dagger A_a+a_sA_a{}^\dagger A_c]\\&+\sum_\sigma g_\sigma[a_\sigma{}^\dagger A_d{}^\dagger A_b+a_\sigma A_b{}^\dagger A_d]\\=&H_0+H_0{}^s+H_0{}^\sigma+\sum_\alpha H_0{}^\alpha+V+\sum_s V^s+\sum_\sigma V^\sigma,\end{aligned}\tag{I.2}$$

where $a^\dagger$, a; $a_s{}^\dagger$, a_s; $a_\sigma{}^\dagger$, a_σ are the emission and ab-

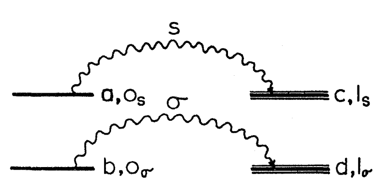

FIG. 8. Level scheme indicating the decay of state $|a,\{0\cdots0_s\cdots\}\rangle$ to $|c,\{0\cdots1_s\cdots\}\rangle$ and state $|b,\{0\cdots0_\sigma\cdots\}\rangle$ to $|d,\{0\cdots1_\sigma\cdots\}\rangle$.

[47] M. Scully, W. E. Lamb, Jr., and M. Sargent III, in Proceedings of the Fourth International Conference on Quantum Electronics, Phoenix, Arizona, 1966 (unpublished). See also Ref. 45.

[48] M. Lax has given a nice discussion of this point in Appendix A of Phys. Rev. **145,** 110 (1966). Our equations (I.14) correspond to Eqs. (A.10) of that paper.

sorption operators for the laser radiation and those of type s and σ, respectively, while the atomic operators $A_a^\dagger$, A_a; $A_b^\dagger$, A_b; $A_c^\dagger$, A_c; $A_d^\dagger$, A_d create and annihilate the atom in states a, b, c, and d. The coupling strengths between the fields and the atom are represented by g, g_s, and g_σ.

The equation of motion in the interaction picture for the density matrix of the atom-field system is

$$\dot{\rho}(t)=-i[V(t),\rho]-i[\sum_s V^s(t)+\sum_\sigma V^\sigma(t),\rho]. \quad \text{(I.3)}$$

Let us introduce the notation

$$(a,n,0_s)=\alpha,\qquad (b,n+1,0_\sigma)=\beta,$$
$$(c,n,1_s)=\gamma,\qquad (d,n+1,1_\sigma)=\delta, \quad \text{(I.4)}$$

with the conventions that a prime on α, β, or δ means that n should be replaced by n' in the definitions of (I.4). We are primarily interested in the quantities $\rho_{\alpha,\alpha'}$, $\rho_{\beta,\beta'}$, $\rho_{\beta,\alpha'}$, $\rho_{\gamma,\gamma'}$, $\rho_{\delta,\delta'}$, which according to (I.3) obey the differential equations

$$\dot{\rho}_{\alpha,\alpha'}=-i[V,\rho]_{\alpha,\alpha'}-i\sum_s[V_{\alpha,\gamma}{}^s\rho_{\gamma,\alpha'}-\rho_{\alpha,\gamma'}V_{\gamma',\alpha'}{}^s], \quad \text{(I.5a)}$$

$$\dot{\rho}_{\beta,\beta'}=-i[V,\rho]_{\beta,\beta'}-i\sum_\sigma[V_{\beta,\delta}{}^\sigma\rho_{\delta,\beta'}-\rho_{\beta,\delta}V_{\delta',\beta'}{}^\sigma], \quad \text{(I.5b)}$$

$$\dot{\rho}_{\beta,\alpha'}=-i[V,\rho]_{\beta,\alpha'}-i[\sum_\sigma V_{\beta,\delta}{}^\sigma\rho_{\delta,\alpha'}-\sum_s\rho_{\beta,\gamma'}V_{\gamma',\alpha'}{}^s], \quad \text{(I.5c)}$$

$$\dot{\rho}_{\gamma,\gamma'}=-i[V_{\gamma,\alpha}{}^s\,\rho_{\alpha,\gamma'}-\rho_{\gamma,\alpha'}V_{\alpha',\gamma'}{}^s], \quad \text{(I.5d)}$$

$$\dot{\rho}_{\delta,\delta'}=-i[V_{\delta,\beta}{}^\sigma\rho_{\beta,\delta'}-\rho_{\delta,\beta'}V_{\beta',\delta'}{}^\sigma]. \quad \text{(I.5e)}$$

We next calculate the effects of the decay modes on our four-level atom by solving for $\rho_{\gamma,\alpha'}$, $\rho_{\delta,\beta'}$, $\rho_{\delta,\alpha'}$, $\rho_{\beta,\gamma'}$, etc. For example, $\rho_{\gamma,\alpha'}$ obeys the differential equation

$$\dot{\rho}_{\gamma,\alpha'}=-i[V(t')_{\gamma,\alpha}{}^s\rho(t')_{\alpha,\alpha'}-\rho(t')_{\gamma,\gamma'}V(t')_{\gamma',\alpha'}{}^s]. \quad \text{(I.5f)}$$

We then have

$$\rho_{\gamma,\alpha'}=-i\int_{t_0}^t[V(t')_{\gamma,\alpha}{}^s\rho(t')_{\alpha,\alpha'}-\rho(t')_{\gamma,\gamma'}V(t')_{\gamma',\alpha'}{}^s]dt', \quad \text{(I.6a)}$$

$$\rho_{\delta,\beta'}=-i\int_{t_0}^t[V_{\delta,\beta}{}^\sigma\rho_{\beta,\beta'}-\rho_{\delta,\delta'}V_{\delta',\beta'}{}^s]dt', \quad \text{(I.6b)}$$

$$\rho_{\delta\alpha'}=-i\int_{t_0}^t[V_{\delta,\beta}{}^\sigma\rho_{\beta,\alpha'}-\rho_{\delta,\gamma'}V_{\gamma'\alpha'}{}^s]dt', \quad \text{(I.6c)}$$

$$\rho_{\beta\gamma'}=-i\int_{t_0}^t[V_{\beta,\delta}{}^\sigma\rho_{\delta,\gamma'}-\rho_{\beta,\alpha'}V_{\alpha',\gamma'}{}^s]dt'. \quad \text{(I.6d)}$$

Since the coupling between the $\{s,\sigma\}$ reservoir and the atom is weak, we assume that for the evaluation of Eqs. (I.6) we may factor the decay radiation density matrix from that of the atom laser, e.g., in Eqs. (I.6),

$$\rho_{\alpha,\alpha'}(t')=\rho_{an0,an'0}(t')\approx\rho_{an,an'}(t')\rho_{0,0}(t'), \quad \text{(I.7)}$$

$$\rho_{\gamma,\gamma'}(t')=\rho_{cn1,cn'1}(t')\approx\rho_{cn,cn'}(t')\rho_{1,1}(t'). \quad \text{(I.8)}$$

Further, if the number of decay modes is large, then to a good approximation

$$\rho_{0,0}(t')\approx\rho_{0,0}(t_0)=1, \quad \text{(I.9a)}$$

$$\rho_{1,1}(t')\approx\rho_{1,1}(t_0)=0. \quad \text{(I.9b)}$$

Then Eqs. (I.6) become

$$\rho_{\gamma,\alpha'}(t)=-i\int_{t_0}^t dt'V_{\gamma,\alpha}{}^s(t')\rho_{an,an'}(t'), \quad \text{(I.10a)}$$

$$\rho_{\delta,\beta'}(t)=-i\int_{t_0}^t dt'V_{\delta,\beta}{}^\sigma(t')\rho_{bn+1,bn'+1}(t'), \quad \text{(I.10b)}$$

$$\rho_{\delta,\alpha'}(t)=-i\int_{t_0}^t dt'V_{\delta,\beta'}{}^\sigma(t')\rho_{bn+1,an'}(t'), \quad \text{(I.10c)}$$

$$\rho_{\beta,\gamma'}(t)=-i\int_{t_0}^t dt'V_{\alpha,\gamma'}{}^s(t')\rho_{bn+1,an'}(t'). \quad \text{(I.10d)}$$

Substituting Eqs. (I.10) into Eqs. (I.5) and tracing over $\{s\}$ and $\{\sigma\}$ we find

$$\dot{\rho}_{an,an'}=-i[V(t),\rho]_{an,an'}-\mathrm{Tr}_{\{s\sigma\}}\left[\sum_s V_{\alpha,\gamma}{}^s(t)\int_{t_0}^t V_{\gamma,\alpha}{}^s(t')\rho_{an,an'}(t')dt'+\text{etc.}\right], \quad \text{(I.11a)}$$

$$\dot{\rho}_{bn+1,bn'+1}=-i[V(t),\rho]_{bn+1,bn'+1}-\mathrm{Tr}_{\{s,\sigma\}}\left[\sum_\sigma V_{\beta,\delta}{}^\sigma(t)\int_{t_0}^t dt'V_{\delta,\beta}{}^\sigma(t')\rho_{bn+1,bn'+1}(t')+\text{etc.}\right], \quad \text{(I.11b)}$$

$$\dot{\rho}_{bn+1,an'}=-i[V(t),\rho]_{bn+1,an'}-\mathrm{Tr}_{\{s,\sigma\}}\left[\sum_\sigma V_{\beta,\delta}{}^\sigma(t)\int_{t_0}^t dt'V_{\delta,\beta}{}^\sigma(t')\rho_{bn+1,an'}(t')+\sum_s V_{\gamma',\alpha'}{}^s(t)\int_{t_0}^t dt'V_{\alpha',\gamma'}{}^s\rho_{bn+1,an'}(t')\right], \quad \text{(I.11c)}$$

$$\dot{\rho}_{cn,cn'}=\mathrm{Tr}_{\{s,\sigma\}}\left[V_{\gamma,\alpha}{}^s(t)\int_{t_0}^t dt'V_{\alpha',\gamma'}{}^s(t')\rho_{an,an'}(t')+\text{etc.}\right], \quad \text{(I.11d)}$$

$$\dot{\rho}_{dn+1,dn'+1}=\mathrm{Tr}_{\{s,\sigma\}}\left[V_{\delta,\beta}{}^\sigma(t)\int_{t_0}^t dt'V_{\beta',\delta'}{}^\sigma(t)\rho_{bn+1,bn'+1}+\text{etc.}\right], \quad \text{(I.11e)}$$

where etc. means replace n by n' and take the complex conjugate of the first term. Now we consider the density of modes $W(\nu)$ to be so large that we may replace sums by integrals

$$\sum_s \cdots \rightarrow \int_0^\infty d\nu\, W(\nu)\cdots,$$

$$\sum_s \cdots \rightarrow \int_0^\infty d\nu\, W(\nu)\cdots.$$

Making the usual approximation of the Wigner–Weisskopf theory in which the matrix elements and the density-of-states factor are evaluated at resonance, thereby neglecting level shifts, Eq. (I.11) become

$$\dot\rho_{an,an'}=-i[V(t),\rho]_{an,an'}-|V_{\alpha,\gamma}|^2W(\omega(ac))\int_{t_0}^t dt'\int_0^\infty d\nu\,(\exp\{i[\omega(ac)-\nu](t-t')\}+\text{c.c.})\rho_{an,an'}(t'), \quad \text{(I.12a)}$$

$$\dot\rho_{bn+1,bn'+1}=-i[V(t),\rho]_{bn+1,bn'+1}-|V_{\beta,\delta}|^2W(\omega(bd))\int_{t_0}^t dt'\int_0^\infty d\nu\,(\exp\{i[\omega(bd)-\nu](t-t')\}+\text{c.c.})\rho_{bn+1,bn'+1}(t'), \quad \text{(I.12b)}$$

$$\dot\rho_{bn+1,an'}=-i[V(t),\rho]_{bn+1,an'}-|V_{\beta,\delta}|^2W(\omega(bd))\int_{t_0}^t dt'\int_0^\infty d\nu\,\exp\{-i[\omega(bd)-\nu](t-t')\}\rho_{bn+1,an'}(t')$$

$$-|V_{\gamma,\alpha}|^2W(\omega(ac))\int_{t_0}^t dt'\int_0^\infty d\nu\,\exp\{i[\omega(ac)-\nu](t-t')\}\rho_{bn+1,an'}(t'), \quad \text{(I.12c)}$$

$$\dot\rho_{cn,cn'}=|V_{\alpha,\gamma}|^2W(\omega(ac))\int_{t_0}^t dt'\int_0^\infty d\nu\,(\exp\{-i[\omega(ac)-\nu](t-t')\}+\text{c.c.})\rho_{an,an'}(t'), \quad \text{(I.12d)}$$

$$\dot\rho_{dn+1,dn'+1}=|V_{\delta,\beta}|^2W(\omega(bd))\int_{t_0}^t dt'\int_0^\infty d\nu\,(\exp\{-i[\omega(bd)-\nu](t-t')\}+\text{c.c.})\rho_{bn+1,bn'+1}(t'). \quad \text{(I.12e)}$$

In the usual way, we may extend the range of integration to $-\infty$ and use the delta function defined by

$$\int_{-\infty}^{\infty} d\nu\,\exp[\pm i(\omega-\nu)(t-t')]=2\pi\delta(t-t'). \quad \text{(I.13)}$$

We may then write Eqs. (I.12) as

$$\dot\rho_{an,an'}=-i[H_0+V,\rho]_{an,an'}-\gamma_a\rho_{an,an'}, \quad \text{(I.14a)}$$

$$\dot\rho_{bn+1,bn'+1}=-i[H_0+V,\rho]_{bn+1,bn'+1}-\gamma_b\rho_{bn+1,bn'+1}, \quad \text{(I.14b)}$$

$$\dot\rho_{bn+1,an'}=-i[H_0+V,\rho]_{bn+1,an}-\gamma_{ab}\rho_{bn+1,an'}, \quad \text{(I.14c)}$$

$$\dot\rho_{cn,cn'}=\gamma_a\rho_{an,an'}, \quad \text{(I.14d)}$$

$$\dot\rho_{dn+1,dn'+1}=\gamma_b\rho_{bn+1,bn'+1}, \quad \text{(I.14e)}$$

where we have transformed back into the Schrödinger picture as it is best suited for the analysis of Sec. III. The decay constants are given by

$$\gamma_a=2\pi W(\omega(ac))\,|V_{a0,c1}|^2, \quad \text{(I.15)}$$

$$\gamma_b=2\pi W(\omega(bd))\,|V_{b0,d1}|^2, \quad \text{(I.16)}$$

$$\gamma_{ab}=\tfrac{1}{2}(\gamma_a+\gamma_b). \quad \text{(I.17)}$$

Equations (I.14) describe the interaction of the laser field with the a and b atomic states which are decaying to states c and d with the usual spontaneous radiative decay constants γ_a and γ_b.

APPENDIX II: DAMPING OF THE FIELD

To provide our cavity with a finite Q, we here consider a dissipative interaction, equivalent to the Ohmic losses of the semiclassical theory. One can envision several satisfactory dissipation mechanisms: Interaction with random currents, photon-phonon interaction, interaction with a two-level atomic system, etc. As we have developed a machinery for dealing with the latter type of interaction, we will consider the dissipative subsystem to consist of nonresonant two-level atoms, injected at random times t_0 in the lower of the two states $|\alpha\rangle$ and $|\beta\rangle$. The calculation will then follow along the lines of Sec. III.

For an atom injected in the β state

$$\delta\rho_{nn'}=\gamma_\alpha\sigma_{11}+\gamma_\beta\sigma_{22}[n\to n-1,\ n'\to n'-1]-\rho_{nn'}(t_0), \tag{II.1}$$

$$\gamma_\alpha\sigma_{11}=\frac{2\gamma_\alpha\gamma_{\alpha\beta}g^2((n+1)(n'+1))^{1/2}\rho_{n+1,n'+1}(t_0)}{\gamma_\alpha\gamma_\beta(\gamma_{\alpha\beta}^2+\Delta^2)+2g^2\gamma_{\alpha\beta}^2(n+1+n'+1)+g^2(n'-n)[g^2(n'-n)+i\Delta(\gamma_\alpha-\gamma_\beta)]}, \tag{II.2a}$$

$$\gamma_\beta\sigma_{22}=\frac{-i\gamma_\beta[i\gamma_\alpha(\Delta^2+\gamma_{\alpha\beta}^2)+g^2(n'+1)(\Delta+i\gamma_{\alpha\beta})-g^2(n+1)(\Delta-i\gamma_{\alpha\beta})]\rho_{n+1,n'+1}(t_0)}{\gamma_\alpha\gamma_\beta(\gamma_{\alpha\beta}^2+\Delta^2)+2g^2\gamma_{\alpha\beta}^2(n+1+n'+1)+g^2(n'-n)[g^2(n'-n)+i\Delta(\gamma_\alpha-\gamma_\beta)]}. \tag{II.2b}$$

From Eq. (III.1) and (III.2), we find

$$\delta\rho_{nn'}=-\frac{g^2[\gamma_\alpha\gamma_{\alpha\beta}(n+n')+i\gamma_\alpha\Delta(n'-n)+g^2(n'-n)^2]\rho_{n,n'}(t_0)}{\gamma_\alpha\gamma_\beta(\gamma_{\alpha\gamma}^2+\Delta^2)+2\gamma_{\alpha\beta}^2g^2(n+n')+g^2(n'-n)[g^2(n'-n)+i\Delta(\gamma_\alpha-\gamma_\beta)]}$$
$$+\frac{2g^2\{\gamma_\alpha\gamma_{\alpha\beta}[(n+1)(n'+1)]^{1/2}\}\rho_{n+1,n'+1}(t_0)}{\gamma_\alpha\gamma_\beta(\gamma_{\alpha\beta}^2+\Delta^2)+2\gamma_{\alpha\beta}^2g^2(n+1+n'+1)+g^2(n'-n)[g^2(n'-n)+i\Delta(\gamma_\alpha-\gamma_\beta)]}. \tag{II.3}$$

Now we replace $\rho_{nn'}(t_0)\to\rho_{nn'}(t)$ and multiply $\delta\rho_{nn'}$ by r_β to obtain the coarse-grained time derivative representing the effects of damping (cavity Q).

Since dissipation, unlike the laser-atom interaction, is a linear process, we will keep only the lowest-order damping terms. We find that the decay of the laser radiation is described by the expression

$$[d\rho_{nn'}/dt]_{\text{damping}}$$
$$\approx-\{g^2(r_\beta/\gamma_\beta)\gamma_{\alpha\gamma}(n+n')[\gamma_{\alpha\beta}^2+\Delta^2]^{-1}\}\rho_{n,n'}(t)$$
$$+\{2g^2(r_\beta/\gamma_\beta)\gamma_{\alpha\beta}[(n+1)(n'+1)]^{1/2}$$
$$\times[\gamma_{\alpha\beta}^2+\Delta^2]^{-1}\}\rho_{n+1,n'+1}(t). \tag{II.4}$$

We define $C=\nu/Q=2r_\beta(g^2/\gamma_\beta)\gamma_{\alpha\beta}[\gamma_{\alpha\beta}^2+\Delta^2]^{-1}$ and write the damping equation in the form appearing in Sec. III.

$$d\rho_{n,n'}/dt=-\tfrac{1}{2}C(n+n')\rho_{n,n'}$$
$$+C[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1}. \tag{II.5}$$

APPENDIX III: SOLUTION OF THE OFF-DIAGONAL EQUATIONS

If the laser is far enough above threshold, we expect that the lowest eigenvalue will be small and that the eigenfunction will be similar to that found for the steady-state diagonal equation (102). Guided by these physical considerations we propose to look for solutions of the off-diagonal equations in the form

$$\rho_{n,n+k}(t)=\Phi_n(k,t)=N_k\left\{\prod_{l=0}^{n}\left[\frac{A-Bl}{C}\right]\prod_{m=0}^{n+k}\left[\frac{A-Bm}{C}\right]\right\}^{1/2}\exp(-\mu_0^{(k)}t), \tag{III.1}$$

where N_k is a constant determined by the initial conditions. We will use for $R_{n,n'}$ not the complicated expression corresponding to (79), but an expression to second order in g^2,

$$R_{n,n'}=r_a(g^2/\gamma_a\gamma_{ab})[1-(g^2/\gamma_a\gamma_b)(\gamma_a(n'+1+n'+1)+\gamma_b(n+1+n'+1))/\gamma_{ab}]. \tag{III.2}$$

The use of this approximate form for $R_{n,n'}$ is the analog of the third-order perturbation expansion of the semiclassical theory. Inserting (III.1) and (III.2) into (85) we find, after some algebra,

$$\dot\Phi_n(k,t)=-\tfrac{1}{8}k^2(\gamma_a/\gamma_{ab})B\Phi_n(k,t)-[A-B(n+1+\tfrac{1}{2}k)](n+1+\tfrac{1}{2}k)\Phi_n(k,t)$$
$$+[A-B(n+\tfrac{1}{2}k)][n(n+k)]^{1/2}\Phi_{n-1}(k,t)-C(n+\tfrac{1}{2}k)\Phi_n(k,t)+C[(n+1)(n+1+k)]^{1/2}\Phi_{n+1}(k,t). \tag{III.3}$$

From Eqs. (III.1) we may write

$$\Phi_{n+1}(k,t)=\{[A-B(n+1)]^{1/2}[A-B(n+1+k)]^{1/2}/C\}\Phi_n(k,t), \tag{III.4a}$$

$$\Phi_{n-1}(k,t)=\{[A-Bn]^{-1/2}[A-B(n+k)]^{-1/2}\}C\Phi_n(k,t). \tag{III.4b}$$

Since $n \gg k$, we may write (III.4) to a very good approximation as

$$\Phi_{n+1}(k,t)=[\{(A-B(n+1))/C\}-\tfrac{1}{2}(Bk/C)]\Phi_n(k,t), \tag{III.5a}$$

$$\Phi_{n-1}(k,t)=[\{C/(A-Bn)\}+\tfrac{1}{2}\{BkC/(A-Bn)^2\}]\Phi_n(k,t); \tag{III.5b}$$

likewise the radicals $[n(n+k)]^{1/2}$, etc. appearing in Eq. (III.3) may be approximated by

$$[n(n+k)]^{1/2}\approx n+\tfrac{1}{2}k-\tfrac{1}{8}(k^2/n), \tag{III.6a}$$

$$[(n+1)(n+1+k)]^{1/2}\approx n+1+\tfrac{1}{2}k-\tfrac{1}{8}\{k^2/(n+1)\}. \tag{III.6b}$$

Making use of (III.5) and (III.6), (III.3) becomes

$$\begin{aligned}\dot{\Phi}_n(k,t)=&-\tfrac{1}{8}k^2(\gamma_a/\gamma_{ab})B\Phi_n(k,t)-[\tfrac{1}{8}Ck^2\Phi_{n+1}/(n+1)+\tfrac{1}{8}[A-B(n+\tfrac{1}{2}k)]k^2\Phi_{n-1}/n]\\&-[A-B(n+1+\tfrac{1}{2}k)](n+1+\tfrac{1}{2}k)\Phi_n+C[(A-B(n+1))/C-\tfrac{1}{2}Bk/C](n+1+\tfrac{1}{2}k)\Phi_n\\&+\{[A-B(n+\tfrac{1}{2}k)](n+\tfrac{1}{2}k)[C/(A-Bn)+\tfrac{1}{2}BkC/(A-Bn)^2]\}\Phi_n-C(n+\tfrac{1}{2}k)\Phi_n.\end{aligned} \tag{III.7}$$

Neglecting terms involving $\langle n\rangle^{-2}$, Eqs. (III.7) become

$$\dot{\Phi}_n(k,t)=-\mu_0^{(k)}\Phi_n=-\{\tfrac{1}{8}Ck^2/\langle n\rangle+\tfrac{1}{8}(A-B\langle n\rangle)k^2/\langle n\rangle+\tfrac{1}{4}k^2[(A-C)/C]B+\tfrac{1}{8}k^2(\gamma_a/\gamma_{ab})B\}\Phi_n. \tag{III.8}$$

Noting that $(A-B\langle n\rangle)$ is C and that

$$\tfrac{1}{8}k^2\gamma_aB/\gamma_{ab}=\tfrac{1}{4}k^2B-\tfrac{1}{8}k^2\gamma_bB/\gamma_{ab}, \tag{III.9}$$

we may write $\mu_0^{(k)}$ as

$$\mu_0^{(k)}=\tfrac{1}{4}(Ck^2/\langle n\rangle)+\tfrac{1}{4}k^2B[(1-\tfrac{1}{2}(\gamma_b/\gamma_{ab})+(A-C)/C)]. \tag{III.10}$$

The leading term corresponds to

$$D=\tfrac{1}{2}(\nu/Q)\langle n\rangle^{-1}, \tag{III.11}$$

as given in Sec. V.