PHY 742 Quantum Mechanics II
1-1:50 AM MWF Olin 103

Plan for Lecture 2

Quantum particle interacting with classical electromagnetic fields
Reading: Chapter 9 in Carlson's textbook
a. Summary of basic equations
b. Examples based on electrostatic fields

1

PHY 742 Quantum Mechanics II>

MWF 1-1:50 PM	OPL 103	http://www.wfu.edu/~natalie/s20phy742/

Instructor: Natalie Holzwarth	Phone:758-5510	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule for Spring 2020

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	Reading	Topic	HW	Due date
1	Mon: 01/13/2020	Chap. 9	Quantum mechanics of electromagnetic forces	#1	01/22/2020
2	Wed: 01/15/2020	Chap. 9	Quantum mechanics of particle in electrostatic field	#2	01/24/2020
3	Fri: 01/17/2020	Chap. 9			
	Mon: 01/20/2020	No class	Martin Luther King Holiday		
4	Wed: 01/22/2020				
5	Fri: 01/24/2020				
6	Mon: 01/27/2020				
7	Wed: 01/29/2020				
8	Fri: 01/31/2020				

2

General formalism for the quantum mechanics of a particle of mass *m* and charge *q* interacting with a electromagnetic field with scalar potential *U*(r,*t*) and vector potential A(r,*t*) --

$$\text{Recap:}\quad i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=H(\mathbf{r},t)\Psi(\mathbf{r},t)\quad\text{and}\quad i\hbar\frac{\partial\Psi'(\mathbf{r},t)}{\partial t}=H'(\mathbf{r},t)\Psi'(\mathbf{r},t)$$

$$\text{where}\quad H(\mathbf{r},t)=\frac{1}{2m}\left(-i\hbar\nabla-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU(\mathbf{r},t)$$

$$\text{and}\quad H'(\mathbf{r},t)=\frac{1}{2m}\left(-i\hbar\nabla-q\mathbf{A}'(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU'(\mathbf{r},t)$$

$$\mathbf{A}'(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)+\nabla\chi(\mathbf{r},t)\qquad U'(\mathbf{r},t)=U(\mathbf{r},t)-\frac{\partial\chi(\mathbf{r},t)}{\partial t}$$

$$\Rightarrow\Psi'(\mathbf{r},t)=e^{iq\chi(\mathbf{r},t)/\hbar}\Psi(\mathbf{r},t)$$

←**differs only by phase factor**

3

For electrostatic and/or magnetostatic fields, the time dependence of the fields becomes trivial, and we expect stationary state solutions to the Schrödinger equation

Hamiltonian: $H(\mathbf{r}) = \frac{1}{2m}\left(-i\hbar\nabla - q\mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}) + qU(\mathbf{r})$

Time-dependent Schrödinger Eq: $i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = H(\mathbf{r})\Psi(\mathbf{r},t)$

Stationary state solution at energy E: $\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-iEt/\hbar}$

Time-independent Schrödinger Eq: $E\psi(\mathbf{r}) = H(\mathbf{r})\psi(\mathbf{r})$

4

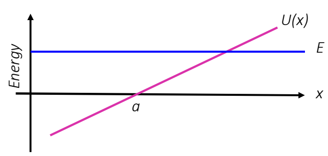
Example of particle interacting with an electromagnetic field

Consider a one-dimensional electrostatic field $\mathbf{E}(\mathbf{r},t) = -F\hat{\mathbf{x}}$

$V(\mathbf{r}) = 0 \quad \mathbf{A}(\mathbf{r},t) = 0 \qquad U(\mathbf{r},t) = U(x) = F(x-a)$

For this case, the stationary state Schrödinger equation at energy E is:

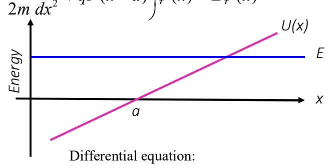
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + qF(x-a)\right)\psi(x) = E\psi(x)$$



5

One dimensional Schrödinger equation for charged particle in an electrostatic field – continued:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + qF(x-a)\right)\psi(x) = E\psi(x)$$



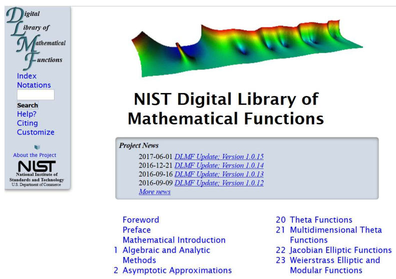
Differential equation:

$$\left(\frac{d^2}{dx^2} - \frac{2mqF}{\hbar^2}(x-b)\right)\psi(x) = 0 \qquad \text{where } b \equiv a + \frac{E}{qF}$$

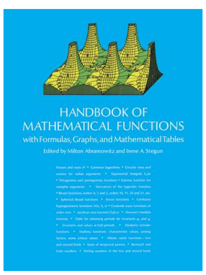
$$\left(\frac{d^2}{du^2} - \alpha u\right)\psi(u) = 0 \qquad \text{where } u \equiv x - b \qquad \alpha \equiv \frac{2mqF}{\hbar^2}$$

6

Digression – library of solutions to differential equations
http://dlmf.nist.gov/



7



http://store.doverpublications.com/0486612724.html

8

§9.2(i) Airy's Equation

9.2.1 $$\frac{d^2w}{dz^2} = zw.$$

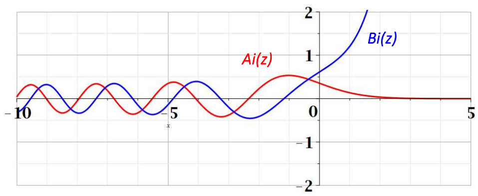
All solutions are entire functions of z.

Standard solutions are:

9.2.2 $$w = \mathrm{Ai}(z),\ \mathrm{Bi}(z),\ \mathrm{Ai}(z e^{\mp 2\pi i/3}).$$

9

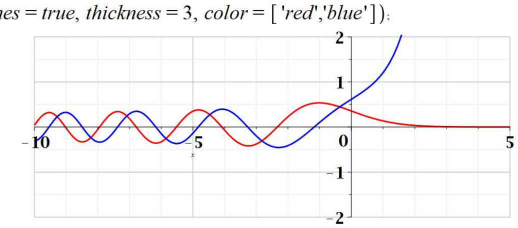
$$\frac{d^2w}{dz^2} = zw$$



10

Example Maple input --

> $plot(\{\text{AiryAi}(x), \text{AiryBi}(x)\}, x=-10..5, -2..2, font=['Times','bold', 24], gridlines=true, thickness=3, color=['red','blue'])$;



11

Differential equation:

$$\left(\frac{d^2}{dx^2} - \frac{2mqF}{\hbar^2}(x-b)\right)\psi(x) = 0 \quad \text{where } b \equiv a + \frac{E}{qF}$$

$$\left(\frac{d^2}{du^2} - \alpha u\right)\psi(u) = 0 \quad \text{where } u \equiv x - b \quad \alpha \equiv \frac{2mqF}{\hbar^2}$$

Airy's equation

$$\left(\frac{d^2}{dz^2} - z\right)Ai(z) = 0$$

Note that the Schroedinger equation can be multiplied by a constant:

$$C\left(\frac{d^2}{du^2} - \alpha u\right)\psi(u) = 0$$

Changing variables: $z = C\alpha u$

$$C\frac{d^2}{du^2} = C^3\alpha^2\frac{d^2}{dz^2} \quad \Rightarrow C = \alpha^{-2/3} \quad \Rightarrow z = \alpha^{1/3}u$$

$$\Rightarrow \psi(u) = \mathcal{N} Ai(\alpha^{1/3}u)$$

normalization constant

12

Some properties of Airy functions –

Integral form:

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right)\,\mathrm{d}t\,.$$

Behavior as $z \to \infty$

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-\frac{2}{3}z^{3/2}}$$

Behavior as $-z \to \infty$

$$Ai(-z) \approx \frac{1}{\sqrt{\pi}z^{1/4}}\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)$$

13

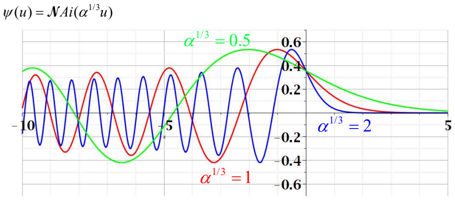
Summary of results

Differential equation:

$$\left(\frac{d^2}{dx^2} - \frac{2mqF}{\hbar^2}(x-b)\right)\psi(x) = 0 \quad \text{where } b \equiv a + \frac{E}{qF}$$

$$\left(\frac{d^2}{du^2} - \alpha u\right)\psi(u) = 0 \quad \text{where } u \equiv x - b \quad \alpha \equiv \frac{2mqF}{\hbar^2}$$

$$\psi(u) = \mathcal{N}Ai(\alpha^{1/3}u)$$



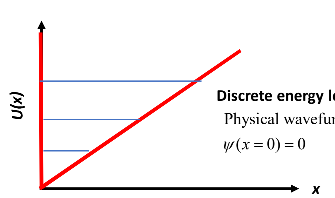
Note that in this case, physical solutions exist for all energies *E*; the wavefunction oscillates for *x<a+E/qF* and decays for *x>a+E/qF* .

14

Related example with bound stationary state solutions --

Consider a spatially confined one-dimensional electrostatic field :

$$V(\mathbf{r}) = 0 \quad \mathbf{A}(\mathbf{r},t) = 0 \qquad U(\mathbf{r},t) = U(x) = \begin{cases} \infty & \text{for } x < 0 \\ Fx & \text{for } x > 0 \end{cases}$$



Discrete energy levels

Physical wavefunctions must satisfy

$$\psi(x=0) = 0 \qquad \psi(x \to \infty) = 0$$

15