PHY 742 Quantum Mechanics II
1-1:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Plan for Lecture 24

Interaction of quantum electromagnetic fields with matter

Read Professor Carlson's textbook: Chapter XVIII. Photons and Atoms

1. **Review of quantum theory of electromagnetism**
2. **Quantum treatment of the interaction of atoms and electromagnetic fields**
3. **Examples of atomic transitions**
4. **Some comments on lasers and masers**

This lecture re-examines the interaction of electromagnetic radiation with charged particles, now including the quantum effects of the fields. This material is treated in Professor Carlson's text in Chapter 18 (A&B).

21	Mon: 03/23/2020	Chap. 17	Quantization of the Electromagnetic Field	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 17	Quantization of the Electromagnetic Field	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 17	Quantization of the Electromagnetic Field	#19	03/30/2020
24	Mon: 03/30/2020	Chap. 18	Photons and atoms		
25	Wed: 04/01/2020				
26	Fri: 04/03/2020				
27	Mon: 04/06/2020				
28	Wed: 04/08/2020				
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020				
30	Wed: 04/15/2020				
31	Fri: 04/17/2020				
32	Mon: 04/20/2020				
33	Wed: 04/22/2020				
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

There is not a new homework for this lecture.

Summary of quantum electromagnetism

Previously, we derived the quantum electromagnetic Hamiltonian (omitting diverging term)

$$H_{\text{EM}} = \sum_{\mathbf{k}'\sigma'} \left(\hbar\omega_{\mathbf{k}'} a^\dagger_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} \right)$$

This is expressed in terms of operators $a_{\mathbf{k}\sigma}$ and $a^\dagger_{\mathbf{k}\sigma}$ operators for wavevector $\mathbf{k}$ and polarization σ.

With commutation relations: $\left[a_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'} \right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$ $\left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'} \right] = 0$ $\left[a^\dagger_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'} \right] = 0$

It is convenient to define the photon number operator

$\mathbf{N}_{\mathbf{k}\sigma} \equiv a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$ with eigenvalues/eigenstates $\mathbf{N}_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle = n_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle$

Note that each wavevector $\mathbf{k}$ and polarization σ is independent (separable) in the EM Hamiltonian so that the system eigenstates are products of eigenstates for each mode:

$$\left| n_{\mathbf{k}_1\sigma_1} n_{\mathbf{k}_2\sigma_2} n_{\mathbf{k}_3\sigma_3} n_{\mathbf{k}_4\sigma_4} \ldots \right\rangle = \left| n_{\mathbf{k}_1\sigma_1} \right\rangle \left| n_{\mathbf{k}_2\sigma_2} \right\rangle \left| n_{\mathbf{k}_3\sigma_3} \right\rangle \left| n_{\mathbf{k}_4\sigma_4} \right\rangle \ldots.$$

The is a review of equations discussed in Lecture 22 & 23

Properties of the creation and annihilation operators:

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Quantum mechanical form of vector potential --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Note: We are assuming that the polarization vector is real. More generally there is a phase factor for each mode which we are ignoring at this moment.

Continuing review of previous results.

Quantum mechanical form of **A**, **E**, and **B** fields --

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_\mathbf{k}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t}+a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)}\right)$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\mathbf{E}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_\mathbf{k}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t}-a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)}\right)$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\mathbf{B}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_\mathbf{k}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t}-a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)}\right)$$

Review continued.

Previously (Lecture 12), we considered a charged particle in the presence of a classical electromagnetic field characterized by vector potential A and scalar potential *U:*

Hamiltonian of particle and field: $H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU(\mathbf{r},t)$

Hamiltonian of particle alone: $H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$

First order interaction Hamiltonian: $H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}+\frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right)+qU(\mathbf{r},t)$

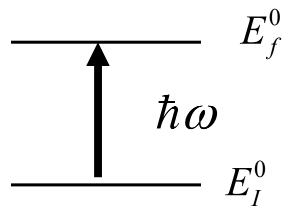
Time dependent electric field: $\mathbf{F}(\mathbf{r},t)=-\nabla U(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$

We used time dependent perturbation theory to analyze the effects of H^1

Now consider the treatment of the interaction of a particle with a classical electromagnetic field as covered in Lecture 12.

Fermi Golden rule for the rate of transition between states *I* and *f*:

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$



Resonant time-dependent perturbation theory lead to Fermi's Golden rule.

What is different about the quantum case?

1. **Minor differences only for cases of very small or large EM fields?**
2. **New physics introduced?**

Please weigh in on this question.

What is different about the quantum case?

Using our quantum treatment, it is convenient to assume that the scalar field $U(\mathbf{r},t)=0$ and $\nabla\cdot\mathbf{A}(\mathbf{r},t)=0$ --

$$H_{\text{EM}}=\sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)$$

Hamiltonian of system: $H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+H_{\text{EM}}$

Hamiltonian of separate particle and EM systems: $H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})+H_{\text{EM}}$

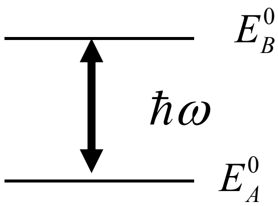
First order interaction Hamiltonian: $H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}$

Time dependent vector potential: $\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

Jumping into the quantum case, we need to modify the previous treatment by adding the Hamiltonian for the quantum electromagnetic field. We also need to use the electromagnetic potentials.

We can still use the Fermi Golden rule for transitions between two states of the zero order system $A^0 \leftarrow\rightarrow B^0$

$$\mathcal{R}_{A^0 \leftrightarrow B^0} \approx \frac{2\pi}{\hbar} \left| \left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle \right|^2 \delta\left(\pm\hbar\omega \mp E_B^0 \pm E_A^0 \right)$$



Now the states $\left|A^0\right\rangle$ and $\left|B^0\right\rangle$ include both the eigenstates of the isolated particle and of the isolated EM system. For example we can denote

$\left|A^0\right\rangle = \left|p_A; n_{\mathbf{k}_A\sigma_A}\right\rangle \qquad \left|B^0\right\rangle = \left|p_B; n_{\mathbf{k}_B\sigma_B}\right\rangle$

In these terms the matrix elements can be evaluated --

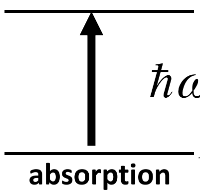
$$\left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \left(\sqrt{n_{\mathbf{k}_A\sigma_A}} \left\langle n_{\mathbf{k}_B\sigma_B} \middle| n_{\mathbf{k}_A\sigma_A} - 1 \right\rangle + \sqrt{n_{\mathbf{k}_A\sigma_A}+1} \left\langle n_{\mathbf{k}_B\sigma_B} \middle| n_{\mathbf{k}_A\sigma_B} + 1 \right\rangle \right)$$

$$\Rightarrow n_{\mathbf{k}_B\sigma_B} = n_{\mathbf{k}_A\sigma_A} \pm 1 \quad \text{corresponding to absorption or emission of a photon}$$

Now the zero order Hamiltonian includes both the particle states and the photon states. In writing the matrix element, we leave the portions involving the particle states as in the classical treatment, but the photon states need also be evaluated.

Some details -- assume $E_B^0 > E_A^0$

$$\mathcal{R}_{A^0 \to B^0} \approx \frac{2\pi}{\hbar} \left| \left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$



$E_B^0 \quad n_{\mathbf{k}_B\sigma_B} = n_{\mathbf{k}_A\sigma_A} - 1$

$\hbar\omega$

$E_A^0 \quad n_{\mathbf{k}_A\sigma_A}$

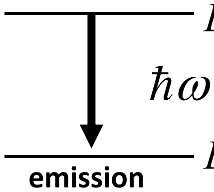
$$\mathcal{R}_{A^0 \to B^0} \approx \frac{2\pi}{\hbar} \left| \frac{q}{m} \sqrt{\frac{\hbar\, n_{\mathbf{k}_A\sigma_A}}{2V\epsilon_0\omega_{\mathbf{k}}}} \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$

$$= \frac{\pi q^2}{m^2 V \epsilon_0 \omega_{\mathbf{k}}} n_{\mathbf{k}_A\sigma_A} \left| \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$

First consider the case of the absorption. We see that the transition rate depends is proportional to the number of photons in the initial state.

More details -- assume $E_B^0 > E_A^0$

$$\mathcal{R}_{B^0 \to A^0} \approx \frac{2\pi}{\hbar} \left| \left\langle A^0 \middle| \tilde{H}^1 \middle| B^0 \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0\right)$$



$E_B^0 \quad n_{\mathbf{k}_B\sigma_B}$

$E_A^0 \quad n_{\mathbf{k}_A\sigma_A} = n_{\mathbf{k}_B\sigma_B} + 1$

$$\mathcal{R}_{B^0 \to A^0} \approx \frac{2\pi}{\hbar} \left| \frac{q}{m} \sqrt{\frac{\hbar\ \left(n_{\mathbf{k}_B\sigma_B} + 1\right)}{2V\epsilon_0\omega_{\mathbf{k}}}} \left\langle p_A \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_B \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0\right)$$

$$= \frac{\pi q^2}{m^2 V \epsilon_0 \omega_{\mathbf{k}}} \left(n_{\mathbf{k}_B\sigma_B} + 1\right) \left| \left\langle p_A \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_B \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0\right)$$

Now consider the case of emission. The transition rate in this case is proportional to the number of photons in the initial state plus 1. What do you suppose is the significance of that.

What is different about the quantum case?

Classical EM field

- **Matrix element depends of atomic selection rules**
- **Matrix element depends on EM field intensity**

Quantum EM field

- **Matrix element depends of atomic selection rules**
- **Matrix element depends on photon eigenstates; absorption different from emission**
- **Possibility of spontaneous emission**

Here are some comments about the differences between the classical and quantum cases.

Lasers and Masers were developed to make use of the relationship between absorption and emission of EM radiation

Rev. Mod. Phys. 99, S263 (1999)

Laser physics: Quantum controversy in action

W. E. Lamb

Optical Sciences Center, University of Arizona, Tucson, Arizona, 85721

W. P. Schleich

Abteilung für Quantenphysik, Albert-Einstein Allee 11, Universität Ulm, D-89069 Ulm, Germany

M. O. Scully

Department of Physics, Texas A&M University, College Station, Texas 77843 and Max-Planck-Institut für Quantenoptik, Hans-Kopfermann Straße 1, D-85748 Garching, Germany

C. H. Townes

Department of Physics, University of California at Berkeley, Berkeley, California 94720

We summarize the history and discuss quantum subtleties of maser/laser physics from early days until the present. [S0034-6861(99)03302-4]

The physics discussed here forms the basis of the laser technology. This is a historical retrospective of some of the ideas used to develop various laser types by some of the key players.

Phys. Rev. 159, 208 (1967)

PHYSICAL REVIEW VOLUME 159, NUMBER 2 10 JULY 1967

Quantum Theory of an Optical Maser.* I. General Theory

MARLAN O. SCULLY† AND WILLIS E. LAMB, JR.
Department of Physics, Yale University, New Haven, Connecticut
(Received 9 February 1967)

A quantum statistical analysis of an optical maser is presented in generalization of the recent semiclassical theory of Lamb. Equations of motion for the density matrix of the quantized electromagnetic field are derived. These equations describe the irreversible dynamics of the laser radiation in all regions of operation (above, below, and at threshold). Nonlinearities play an essential role in this problem. The diagonal equations of motion for the radiation are found to have an apparent physical interpretation. At steady state, these equations may be solved via detailed-balance considerations to yield the photon statistical distribution $\rho_{n,n}$. The resulting distribution has a variance which is significantly larger than that for coherent light. The off-diagonal elements of the radiation density matrix describe the effects of phase diffusion in general and provide the spectral profile $|E(\omega)|^2$ as a special case. A detailed discussion of the physics involved in this paper is given in the concluding sections. The theory of the laser adds another example to the short list of solved problems in irreversible quantum statistical mechanics.

This is one of the early theory developments of laser physics. It is based on coupling the transition rate equations with rate equations for the photon populations in such a way as to achieve large field strengths.