PHY 742 Quantum Mechanics
1-1:50 AM MWF Olin 103

Plan for Lecture 15:

The Dirac equation

Chap. 16 in Professor Carlson's textbook:

1. **Dirac equation for a free particle**
2. **Dirac equation for a hydrogen atom**

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8	Fri: 01/31/2020	Chap. 12	Variational and other approximation methods	#8	02/07/2020
9	Mon: 02/03/2020	Chap. 2,6	Single particle states of molecules and solids	#9	02/10/2020
10	Wed: 02/05/2020	Chap. 2,6	H_2^+ molecular ion; Born Oppenheimer approximation	#10	02/12/2020
11	Fri: 02/07/2020	Chap. 15	Time-dependent perturbations	#11	02/14/2020
12	Mon: 02/10/2020	Chap. 15	Time-dependent perturbations	#12	02/14/2020
13	Wed: 02/12/2020	Chap. 15	Time-dependent perturbations	#13	02/17/2020
14	Fri: 02/14/2020	Chap. 16	The Dirac equation		
15	Mon: 02/17/2020	Chap. 16	The Dirac equation	#14	02/19/2020
16	Wed: 02/19/2020				
17	Fri: 02/21/2020				
18	Mon: 02/24/2020				
19	Wed: 02/26/2020				
20	Fri: 02/28/2020				
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		

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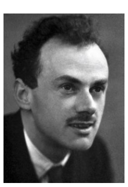

P. A. M Dirac 1902-1984
Won Nobel Prize in Physics in 1933
together with Erwin Schrödinger

The quantum theory of the electron

Paul Adrien Maurice Dirac

Published: **01 February 1928**

https://doi.org/10.1098/rspa.1928.0023

PROCEEDINGS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

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Energy ←→ Momentum relationships in classical and quantum mechanics
Focusing on treatment of Fermi particles

	Non-relativistic mechanics	Relativistic mechanics
Classical	$E=\frac{\mathbf{p}^2}{2m}$	$E^2-\mathbf{p}^2c^2=\left(mc^2\right)^2$ $\Downarrow$ (with some license) $\left(E-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(E+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)=\left(mc^2\right)^2$
	$\Downarrow$	$\Downarrow$
Quantum	$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$	$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi=\left(mc^2\right)^2\Psi$

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Relativistic relationships – continued
Ref: J. J. Sakurai, Advanced Quantum Mechanics

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi=\left(mc^2\right)^2\Psi$$

Let $\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi\equiv mc^2\Psi^R \qquad \Psi\equiv\Psi^L$

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

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Relativistic relationships – continued
Ref: J. J. Sakurai, Advanced Quantum Mechanics

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Dirac's rearrangement: $\varphi^U=\Psi^R+\Psi^L$

$\varphi^L=\Psi^R-\Psi^L$

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\varphi^U \\ \varphi^L\end{pmatrix}=mc^2\begin{pmatrix}\varphi^U \\ \varphi^L\end{pmatrix}$$

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Relativistic relationships – continued

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = mc^2\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Further rearrangements:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$\Downarrow$$

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

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Four component wavefunction of free Fermi particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Assume $\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} \chi^U(\mathbf{k}) \\ \chi^L(\mathbf{k}) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r} - iEt/\hbar}$

$$\Rightarrow \chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E - mc^2}\chi^L(\mathbf{k})$$

$$\chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E + mc^2}\chi^U(\mathbf{k})$$

$$E^2 = \hbar^2c^2\mathbf{k}^2 + m^2c^4$$

$$E = \pm\sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$$

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Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathbf{k}\cdot\boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$ Define: $k_\pm \equiv k_x \pm ik_y$

$$\chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E - mc^2}\chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E + mc^2}\chi^U(\mathbf{k})$$

Positive energy solutions: $E = \sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

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Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

Define: $k_\pm \equiv k_x \pm ik_y$

$$\chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E - mc^2}\chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E + mc^2}\chi^U(\mathbf{k})$$

Negative energy solutions: $E = -\sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E - mc^2} = -\frac{\hbar k_z c}{|E| + mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E - mc^2} = -\frac{\hbar k_\pm c}{|E| + mc^2}$$

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What does this all mean?

Positive energy solutions: $E = \sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

For $\hbar c|\mathbf{k}| \ll mc^2 \qquad E \approx mc^2 + \frac{\hbar^2|\mathbf{k}|^2}{2m}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Dirac equation for free particles – slightly more convenient notation

mc^2 $\quad E = \sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \tilde{\kappa}_z \\ \tilde{\kappa}_+ \end{pmatrix}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \tilde{\kappa}_- \\ -\tilde{\kappa}_z \end{pmatrix}$$

$$\tilde{\kappa}_z \equiv \frac{\hbar k_z c}{|E| + mc^2} \qquad \tilde{\kappa}_\pm \equiv \frac{\hbar k_\pm c}{|E| + mc^2}$$

$-mc^2$ $\quad E = -\sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} -\tilde{\kappa}_z \\ -\tilde{\kappa}_+ \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} -\tilde{\kappa}_- \\ \tilde{\kappa}_z \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For free particle, *E>0* and *E<0* solutions represent distinct physical situations.

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Dirac equation for a free particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c\\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

Other convenient notations in terms of 4×4 matrices

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\quad H=\mathbf{p}\cdot\boldsymbol{\alpha}c+mc^2\beta$$

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad I\equiv I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

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Dirac equation for Fermi particle in a scalar potential field

For free particle: $H=\mathbf{p}\cdot\boldsymbol{\alpha}c+mc^2\beta$

where: $\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}$ $\beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$ and where:

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad I\equiv I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Dirac's suggestion for representing a scalar potential field:

$$H=\mathbf{p}\cdot\boldsymbol{\alpha}c+mc^2\beta+V(\mathbf{r})I_4 \quad \text{where } I_4=\begin{pmatrix}I_2 & 0\\ 0 & I_2\end{pmatrix}$$

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Dirac equation for electron in the field of a H-like ion

$$H=\mathbf{p}\cdot\boldsymbol{\alpha}c+mc^2\beta+V(\mathbf{r})I_4$$

For H-like ion with nuclear charge Z:

$$V(\mathbf{r})=V(r)=-\frac{Ze^2}{r}$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

Stationary state solutions:

$$\Psi(\mathbf{r},t)=\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix}e^{-iEt/\hbar}$$

$$\begin{pmatrix}mc^2+V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c\\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2+V(r)\end{pmatrix}\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix}=E\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix}$$

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Dirac equation for electron in the field of a H-like ion -- continued

$$\underbrace{\begin{pmatrix} mc^2+V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2+V(r)\end{pmatrix}}_{H}\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix} = E\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix}$$

In order to find an efficient functional form for the eigenfunctions, it is helpful to determine the constants of the motion:

Total angular momentum: $\mathbf{J}=\mathbf{L}+\frac{\hbar}{2}\boldsymbol{\sigma}$ in terms of J_z and $\mathbf{J}^2$

Special K operator: $K=\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar I_2 & 0\\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L}-\hbar I_2\end{pmatrix}$

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Digression – useful identity for operator vectors **A** and **B**

$$(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\begin{pmatrix} A_z & A_x-iA_y\\ A_x+iA_y & -A_z\end{pmatrix}\begin{pmatrix} B_z & B_x-iB_y\\ B_x+iB_y & -B_z\end{pmatrix} = \begin{pmatrix}\mathbf{A}\cdot\mathbf{B}+iC_z & iC_x+C_y\\ iC_x-C_y & \mathbf{A}\cdot\mathbf{B}-iC_z\end{pmatrix}$$

where

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$ and $\mathbf{C}=\mathbf{A}\times\mathbf{B}$

$C_x = A_yB_z - A_zB_y \quad C_y = A_zB_x - A_xB_z \quad C_z = A_xB_y - A_yB_x$

Note that: $\frac{(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{r})}{r^2} = I_2$

It follows that: $(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{p}) = \frac{(\boldsymbol{\sigma}\cdot\mathbf{r})}{r^2}(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{p}) = \frac{(\boldsymbol{\sigma}\cdot\mathbf{r})}{r^2}\left(-i\hbar r\frac{\partial}{\partial r} + i\boldsymbol{\sigma}\cdot\mathbf{L}\right)$

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Dirac equation for electron in the field of a H-like ion -- continued

$$H=\begin{pmatrix} mc^2+V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2+V(r)\end{pmatrix}$$

$$J_z=\begin{pmatrix} L_z+\frac{1}{2}\hbar\sigma_z & 0\\ 0 & L_z+\frac{1}{2}\hbar\sigma_z\end{pmatrix}$$

$$\mathbf{J}^2=\begin{pmatrix} (\mathbf{L}+\frac{1}{2}\hbar\boldsymbol{\sigma})^2 & 0\\ 0 & (\mathbf{L}+\frac{1}{2}\hbar\boldsymbol{\sigma})^2\end{pmatrix} = \begin{pmatrix}\mathbf{L}^2+\frac{3\hbar^2}{4}I_2+\hbar\boldsymbol{\sigma}\cdot\mathbf{L} & 0\\ 0 & \mathbf{L}^2+\frac{3\hbar^2}{4}I_2+\hbar\boldsymbol{\sigma}\cdot\mathbf{L}\end{pmatrix}$$

$$K=\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar I_2 & 0\\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L}-\hbar I_2\end{pmatrix}$$

Note that:

$$K^2 = \mathbf{J}^2 + \tfrac{1}{4}\hbar^2$$

Commutation relations:

$[H,\mathbf{J}^2]=0 \quad [K,\mathbf{J}^2]=0 \quad [K,J_z]=0 \quad [H,K]=0,$ etc.

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Dirac equation for electron in the field of a H-like ion -- continued

$$\begin{pmatrix} mc^2+V(r) & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2+V(r) \end{pmatrix}\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = E\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix}$$

We can show that:
$$\begin{pmatrix} \varphi^U(\mathbf{r}) \\ \varphi^L(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} g_{\kappa J}(r)\chi_{\kappa JM}(\hat{\mathbf{r}}) \\ if_{\kappa J}(r)\chi_{-\kappa JM}(\hat{\mathbf{r}}) \end{pmatrix}$$

More details about spin-angular functions:

Eigenfunctions of $\mathbf{J}^2$ and J_z: $\left|JM\right\rangle$

$$\mathbf{J}^2\left|JM\right\rangle = \hbar^2 J(J+1)\left|JM\right\rangle$$

$$J_z\left|JM\right\rangle = \hbar M\left|JM\right\rangle$$

$$K^2\left|JM\right\rangle = \hbar^2\left(J(J+1)+\tfrac{1}{4}\right)\left|JM\right\rangle = \hbar^2\left(J+\tfrac{1}{2}\right)^2\left|JM\right\rangle$$

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Dirac equation for electron in the field of a H-like ion -- continued

Eigenvalues of K operator:

$$K = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L}+\hbar I_2 & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L}-\hbar I_2 \end{pmatrix}$$

$$K\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = -\hbar\kappa\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$K^2\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \hbar^2\left(J+\tfrac{1}{2}\right)^2\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

It can also be shown that φ^U and φ^L are eigenvectors of $\mathbf{L}^2$:

$\mathbf{L}^2\varphi^U = \hbar^2 l_U(l_U+1)\varphi^U$ and $\mathbf{L}^2\varphi^L = \hbar^2 l_L(l_L+1)\varphi^L$

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More relationships between the operators

$$\mathbf{L}^2 = \mathbf{J}^2 - \hbar\boldsymbol{\sigma}\cdot\mathbf{L} - \tfrac{3}{4}\hbar^2 = \mathbf{J}^2 + \tfrac{1}{4}\hbar^2 - \hbar\left(\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar\right)$$

Relationships between the upper and lower component functions:

$\mathbf{L}^2\varphi^U = \hbar^2 l_U(l_U+1)\varphi^U$ and $\mathbf{L}^2\varphi^L = \hbar^2 l_L(l_L+1)\varphi^L$

$\left(\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar\right)\varphi^U = -\kappa\varphi^U$ and $\left(\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar\right)\varphi^L = \kappa\varphi^L$

After some algebra --

$$l_L(l_L+1) = l_U(l_U+1) - 2\kappa$$

$$l_U(l_U+1) = j(j+1) + \kappa + \tfrac{1}{4}$$

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Summary of allowed combinations of eigenvalues

	l_U	l_L
$\kappa = -\left(j+\frac{1}{2}\right)$	$j-\frac{1}{2}$	$j+\frac{1}{2}$
$\kappa = +\left(j+\frac{1}{2}\right)$	$j+\frac{1}{2}$	$j-\frac{1}{2}$

Alternatively

	j	l_L
$\kappa = -\left(l_U+1\right)$	$l_U+\frac{1}{2}$	l_U+1
$\kappa = +l_U$	$l_U-\frac{1}{2}$	l_U-1

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Addition of angular momentum - orbital angular momentum and spin angular momentum

Clebsch-Gordon coefficients

$$\left|JM, j_1 j_2\right\rangle = \sum_{m_1,m_2} \left|j_1 m_1, j_2 m_2\right\rangle \left\langle j_1 m_1, j_2 m_2 \middle| JM, j_1 j_2\right\rangle$$

$j_1 m_1 \Rightarrow l m_l$ $\qquad J = l + \frac{1}{2}$

$j_2 m_2 \Rightarrow \frac{1}{2} m_s$ $\qquad$ or $\; J = l - \frac{1}{2}$

Clebsch-Gordon coefficients for this case:

$j_1 = l \qquad j_2 = s = \frac{1}{2}$

$$\left|\left(l+\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}\left|l\left(M-\tfrac{1}{2}\right);\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}\left|l\left(M+\tfrac{1}{2}\right);\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\left(l-\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = -\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}\left|l\left(M-\tfrac{1}{2}\right);\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}\left|l\left(M+\tfrac{1}{2}\right);\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

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Another representation of the spin-angular function in terms of spherical harmonic functions and eigenvectors of σ_z

$$\left|\left(l+\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}Y_{l(M-\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}Y_{l(M+\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}0\\1\end{pmatrix}$$

$$\left|\left(l-\tfrac{1}{2}\right)M; l\tfrac{1}{2}\right\rangle = -\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}Y_{l(M-\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}Y_{l(M+\frac{1}{2})}(\hat{\mathbf{r}})\begin{pmatrix}0\\1\end{pmatrix}$$

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Consider: $$\left(\boldsymbol{\sigma}\cdot\mathbf{L}+\hbar I_2\right)\left|JM;l\tfrac{1}{2}\right\rangle=\hbar\left(J(J+1)-l(l+1)+\tfrac{1}{4}\right)I_2\left|JM;l\tfrac{1}{2}\right\rangle$$
$$=-\kappa\left|JM;l\tfrac{1}{2}\right\rangle$$

Possibilities:

$J=l+\frac{1}{2}\qquad \kappa=-l-1$

$J=l-\frac{1}{2}\qquad \kappa=l$

Combinations:

κ	j	l
$\kappa=-1$	$j=\frac{1}{2}$	$l=0$
$+1$	$\frac{1}{2}$	1
-2	$\frac{3}{2}$	1
$+2$	$\frac{3}{2}$	2
-3	$\frac{5}{2}$	2
$+3$	$\frac{5}{2}$	3

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$$\begin{pmatrix}\varphi^U(\mathbf{r})\\ \varphi^L(\mathbf{r})\end{pmatrix}=\begin{pmatrix}g_{\kappa J}(r)\chi_{\kappa JM}(\hat{\mathbf{r}})\\ if_{\kappa J}(r)\chi_{-\kappa JM}(\hat{\mathbf{r}})\end{pmatrix}$$

Next time, we will show that the coupled radial equations take the form:

$$\frac{dg}{dr}+\frac{\kappa+1}{r}g=\frac{1}{\hbar c}\left(E+mc^2-V(r)\right)f$$
$$\frac{df}{dr}-\frac{\kappa-1}{r}f=-\frac{1}{\hbar c}\left(E-mc^2-V(r)\right)g$$

Question:
- How do these results reduce to the Schroedinger equation in the non-relativistic limit
- What new physics is contained more generally?

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