# PHY 712 Electrodynamics
## 12-12:50 AM MWF Olin 103

**Plan for Lecture 7:**

**Start reading Chapter 3**

**Solution of Poisson/Laplace equation for special geometries –**

**A. Cylindrical**

**B. Spherical**



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**Course schedule for Spring 2020**

(Preliminary schedule -- subject to frequent adjustment.)

| | Lecture date | JDJ Reading | Topic | HW | Due date |
|---|---|---|---|---|---|
| 1 | Mon: 01/13/2020 | Chap. 1 & Appen. | Introduction, units and Poisson equation | #1 | 01/17/2020 |
| 2 | Wed: 01/15/2020 | Chap. 1 | Electrostatic energy calculations | #2 | 01/22/2020 |
| 3 | Fri: 01/17/2020 | Chap. 1 | Electrostatic potentials and fields | #3 | 01/24/2020 |
| | Mon: 01/20/2020 | No class | Martin Luther King Holiday | | |
| 4 | Wed: 01/22/2020 | Chap. 1 - 3 | Poisson's equation in 2 and 3 dimensions | #4 | 01/27/2020 |
| 5 | Fri: 01/24/2020 | Chap. 1 - 3 | Brief introduction to numerical methods | #5 | 01/31/2020 |
| 6 | Mon: 01/27/2020 | Chap. 2 & 3 | Image charge constructions | #6 | 02/03/2020 |
| 7 | Wed: 01/29/2020 | Chap. 2 & 3 | Cylindrical and spherical geometries | #7 | 02/05/2020 |
| 8 | Fri: 01/31/2020 | Chap. 3 & 4 | Spherical geometry and multipole moments | | |
| 9 | Mon: 02/03/2020 | Chap. 4 | Dipoles and Dielectrics | | |
| 10 | Wed: 02/05/2020 | Chap. 4 | Polarization and Dielectrics | | |
| 11 | Fri: 02/07/2020 | Chap. 5 | Magnetostatics | | |
| 12 | Mon: 02/10/2020 | Chap. 5 | Magnetic dipoles and hyperfine interaction | | |
| 13 | Wed: 02/12/2020 | Chap. 5 | Magnetic dipoles and dipolar fields | | |



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## Colloquium: "First-Principles Multiscale Modelling of Quantum Materials "

Dr. Steve Winter
Junior Project Leader
Institute for Theoretical Physics
Goethe University, Frankfurt
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, January 29, 2020 at 3:00 PM

There will be a reception in Olin Lounge at approximately 4 PM following the colloquium.
All interested persons are cordially invited to attend.

https://www.physics.wfu.edu/events/dr-steve-winter-institute-for-theoretical-physics-frankfurt-germany/



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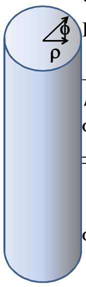
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho,\phi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\phi+\alpha_m)$$



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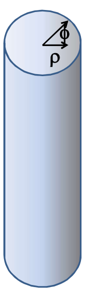
Cylindrical coordinates with trivial *z*-dependence – some details: $\Phi(\rho,\phi)=f(\rho)g(\phi)$




Suppose $\frac{d^2g(\phi)}{d\phi^2}=-m^2g(\phi)$

$$g(\phi)=\sin(m\phi+\alpha_m)$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df_m(\rho)}{d\rho}\right)-\frac{m^2}{\rho^2}f_m(\rho)=0$$

$$f_0(\rho)=\begin{cases}1\\ \ln\rho\end{cases} \qquad f_{m>0}=\rho^{\pm m}$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\varphi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\varphi+\alpha_m)$$



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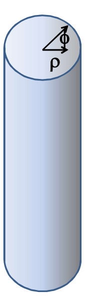
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Green's function appropriate for this geometry with boundary conditions at $\rho=0$ and $\rho=\infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>^{\,2}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos(m(\phi-\phi'))$$



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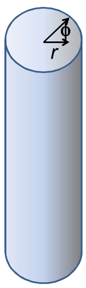
## Comments and details

Change notation

$\rho \Rightarrow r$




$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that: For this extended charge distribution, Coulomb's law in its original form diverges:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$



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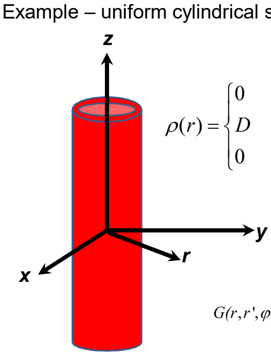
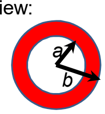
## Example – uniform cylindrical shell:




Top view:




$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\Phi(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\Phi(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi(r,\phi)}{\partial^2\phi}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$



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## Some details

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

In our case: $\Phi(r,\varphi) = \frac{2\pi D}{4\pi\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>^2)\right) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$

For $0 \le r < a$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r')\right)$

For $a \le r < b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\left(\int_a^r r'dr'\left(-\ln(r)\right) + \int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r > b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r)\right)$



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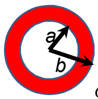
Example continued -- *m=0* only --

Top view:




$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0}\left(b^2 - a^2 - b^2\ln(b^2) + a^2\ln(a^2)\right) & 0 < r < a \\ \frac{D}{4\epsilon_0}\left(b^2 - r^2 - b^2\ln(b^2) + a^2\ln(r^2)\right) & a \le r \le b \\ \frac{D}{4\epsilon_0}\left(a^2 - b^2\right)\ln(r^2) & r > b \end{cases}$$



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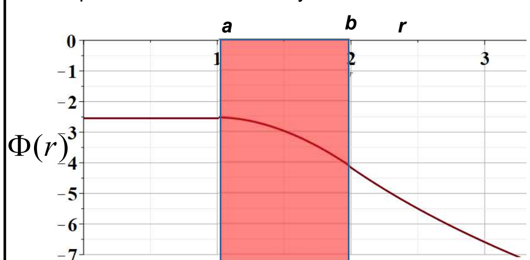
Example continued -- *m=0* only --



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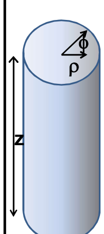
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence




Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$



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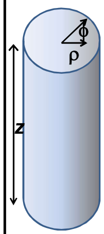
Cylindrical geometry continued:




$$\text{Laplace equation :} \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

One possibility :

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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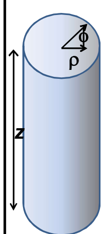
Cylindrical geometry continued:




$$\text{Laplace equation :} \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

Another possibility :

$$\frac{d^2Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

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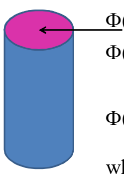
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*

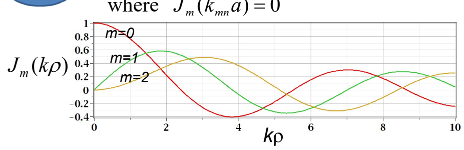


$$\Phi(\rho,\phi,z=L) = V(\rho,\phi)$$

$$\Phi(\rho,\phi,z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi + \alpha_{mn})$$

$$\text{where} \quad J_m(k_{mn}a) = 0$$

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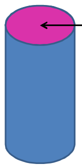
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$
$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of $\varphi$ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^a \rho d\rho J_m^{\,2}(k_{mn}\rho)}$$



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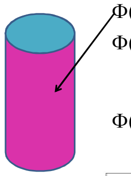
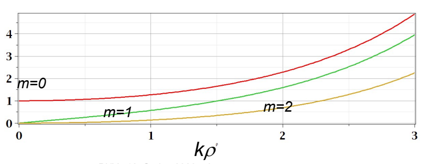
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho$=*a*



$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$
$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn}I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$

$I_m(k\rho)$



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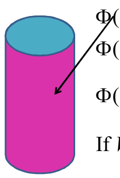
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho$=*a*



$$\Phi(\rho=a,\varphi,z)=V(\varphi,z)$$
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$
$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn}I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(z,\varphi)$ is an even function of $\varphi$ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^L dz\sin\left(\frac{n\pi z}{L}\right)V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^L dz\sin^2\left(\frac{n\pi z}{L}\right)}$$



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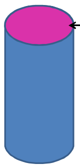
Green's function for Dirchelet boundary value inside cylindar:



$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$

$\Phi(\rho=a,\phi,z)=0\,,\;\Phi(\rho,\phi,z=0)=0$

Expansion in terms of Bessel function zeros: $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=$$
$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\rho' d\rho' dz' G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$
$$+\frac{1}{4\pi}\int_{S;z'=L} d\phi'\rho' d\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$



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Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$




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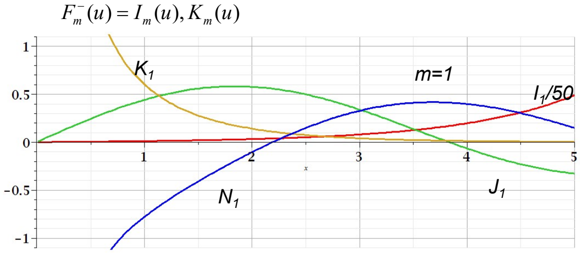
Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



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Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: $x_{mn}$; $\quad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a)J_m(x_{mn'}\rho/a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$

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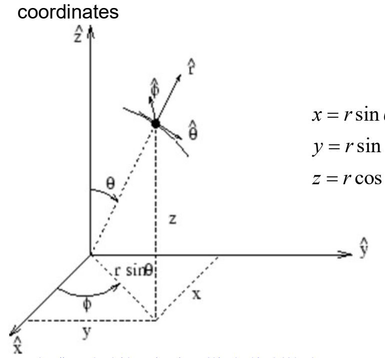
Poisson and Laplace equation in spherical polar coordinates



$$x=r\sin\theta\cos\varphi$$
$$y=r\sin\theta\sin\varphi$$
$$z=r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

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Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\phi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)\Phi=0$$

$$\Phi(r,\theta,\phi)=\sum_{lm}R_{lm}(r)Y_{lm}(\theta,\phi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)Y_{lm}(\theta,\phi)=-l(l+1)Y_{lm}(\theta,\phi)$$

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## Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi)=(-1)^m Y_{l(-m)}^*(\theta,\phi) \quad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi)Y_{l'm'}^*(\theta,\varphi)\equiv\int \sin\theta\, d\theta\, d\varphi\, Y_{lm}(\theta,\varphi)Y_{l'm'}^*(\theta,\varphi)=\delta_{ll'}\delta_{mm'}$$

Completeness:

$$\sum_{lm} Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')=\delta(\hat{\mathbf{r}}-\hat{\mathbf{r}}')\equiv\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')$$

Relationship to Legendre polynomials:

$$Y_{l0}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$



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## Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r\to\infty$:

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$



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## Example -- continued

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose: $\rho(\mathbf{r}')=\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$

$$\int d\Omega'\, Y_{lm}^*(\theta',\varphi')=\sqrt{4\pi}\,\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r})=\frac{4\pi}{4\pi\varepsilon_0}\int_0^\infty r'^2\,dr'\frac{r_<^0}{r_>^1}\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$=\frac{Q}{4\pi\varepsilon_0}\frac{\operatorname{erf}(r/a)}{r}$$



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Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2-2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}}=$$

$$=\frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'+\left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2-\frac{1}{2}\right)+\cdots\right)$$

$$=\frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$



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Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{r}})\,Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}}=\hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=1$

$$\Rightarrow\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|Y_{lm}(\hat{\mathbf{r}})|^2=1$$



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Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}})=\frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}})=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}})=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm2i\phi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}})=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$



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