PHY 712 Electrodynamics

12-12:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Extra notes for Lecture 28:

Continue reading Chap. 14 – Synchrotron radiation

1. **Radiation from electron synchrotron devices**
2. **Radiation from astronomical objects in circular orbits**

In this lecture, we will examine in detail, the angular and spectral properties of synchrotron radiation which is presented in Chapter 14 of Jackson.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources	#19	03/30/2020
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity	#20	04/03/2020
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity	#21	04/06/2020
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles	#22	04/08/2020
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

Outstanding homework and project topics due Mon. 4/13/2020.

Please note due date.

Your questions

From Trevor

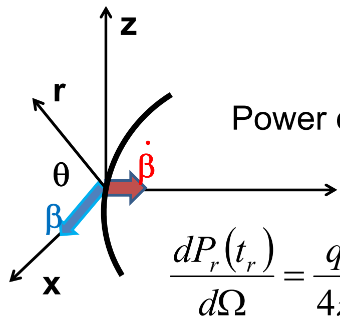
1. On slide, are the assumptions and the integration by parts the steps that make the denominator (1 - beta*rhat)^2 go away? Is this process also helped by the expression given at the top of that same slide, involving d/dt_r?

Comment – perhaps this refers to slide 11 of the original lecture. Some extra steps will be offered when we get to that point of the lecture.

More general comments and pep talk

Slides from original lecture --

Radiation from charged particle in circular path



Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \left. \frac{\left| \dot{\boldsymbol{\beta}} \right|^2 \left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2 - \left(\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}} \right)^2 \left(1 - \beta^2 \right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} \left| \dot{\mathbf{v}} \right|^2 \gamma^4$$

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A convenient geometry to discuss synchrotron radiation of charged particle moving in the x-y plane.

Spectral composition of electromagnetic radiation

Starting with the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

$$\equiv \left|\boldsymbol{a}(t)\right|^2$$

where $$\boldsymbol{a}(t)\equiv\sqrt{\frac{q^2}{4\pi c}}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\right|_{t_r=t-R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Equations for analyzing spectral decomposition of radiation.

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle using Parseval's theorem:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

In terms of the amplitude and its Fourier transform:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \boldsymbol{a}(t)\, e^{i\omega t} \qquad \boldsymbol{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \tilde{\boldsymbol{a}}(\omega)\, e^{-i\omega t}$$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \left|\tilde{\boldsymbol{a}}(\omega)\right|^2 = \int_{0}^{\infty} d\omega \left(\left|\tilde{\boldsymbol{a}}(\omega)\right|^2 + \left|\tilde{\boldsymbol{a}}(-\omega)\right|^2\right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Review of Parceval's theorem and derivation of the spectral expression expressions that we will use.

Spectral composition of electromagnetic radiation -- continued

For our case: $$\boldsymbol{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \boldsymbol{a}(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Basic equations that need to be evaluated for each given particle trajectory.

Spectral composition of electromagnetic radiation -- continued

Evaluating the Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ \boldsymbol{a}(t)\ e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r\, \frac{dt}{dt_r} e^{i\omega(t_r + R(t_r)/c)} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r e^{i\omega(t_r + R(t_r)/c)} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2} \right|_{t_r = t - R/c}$$

Further evaluations.

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathcal{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r e^{i\omega(t_r + R(t_r)/c)} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

Some approximations:

For $r >> R_q(t_r) \qquad R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Up to now, the equations have been exact. Now we consider some reasonable approximations that are appropriate for measuring the spectrum far from the source.

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\boldsymbol{a}}(\omega)=\sqrt{\frac{q^2}{8\pi^2 c}}\int_{-\infty}^{\infty} dt_r\, e^{i\omega\left(t_r+R(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2}\right|_{t_r=t-R/c}$$

Approximate expression:

$$\tilde{\boldsymbol{a}}(\omega)=\sqrt{\frac{q^2}{8\pi^2 c}}e^{i\omega(r/c)}\int_{-\infty}^{\infty} dt_r\, e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\;\; e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c}\right|^2$$

Some details of the approximate equations.

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}=\frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

Integration by parts and assumptions about behaviors at the integration limit, shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}dt_r\ e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\ \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]\right|^2$$

From the tope identity, the intensity expression can be further simplified.

Some details --

Spectral intensity expression that needs to be evaluated:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r = t - R/c}\right|^2$$

It can be shown that: $\dfrac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2} = \dfrac{d}{dt_r}\left(\dfrac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$

$$\int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\Bigg|_{t_r = t-R/c} = \int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)$$

$$= \int_{-\infty}^{\infty} dt_r\ \frac{d}{dt_r}\left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)\right) - i\omega\int_{-\infty}^{\infty} dt_r\ \left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\left(\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right)\right)$$

0

Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \; \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]\right|^2$$

Recall that the spectral intensity is related to the time integrated power:

$$\int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} d\omega \frac{\partial^2 I}{\partial\omega\partial\Omega}$$

Summary of equations that need to be avaluated.

Synchrotron radiation light source installations

Synchrotron at Brookhaven National Lab, NY

E_c= 3 GeV X-ray radiation

https://www.bnl.gov/ps/

Before we start to evaluate the expressions, we will first consider some the large synchrotron radiation installations currently available today. (Actually some of these facilities have shutdown due to the pandemic unless they are being used for studying certain viruses. This is an arial photo of the facility on Long Island, NY.

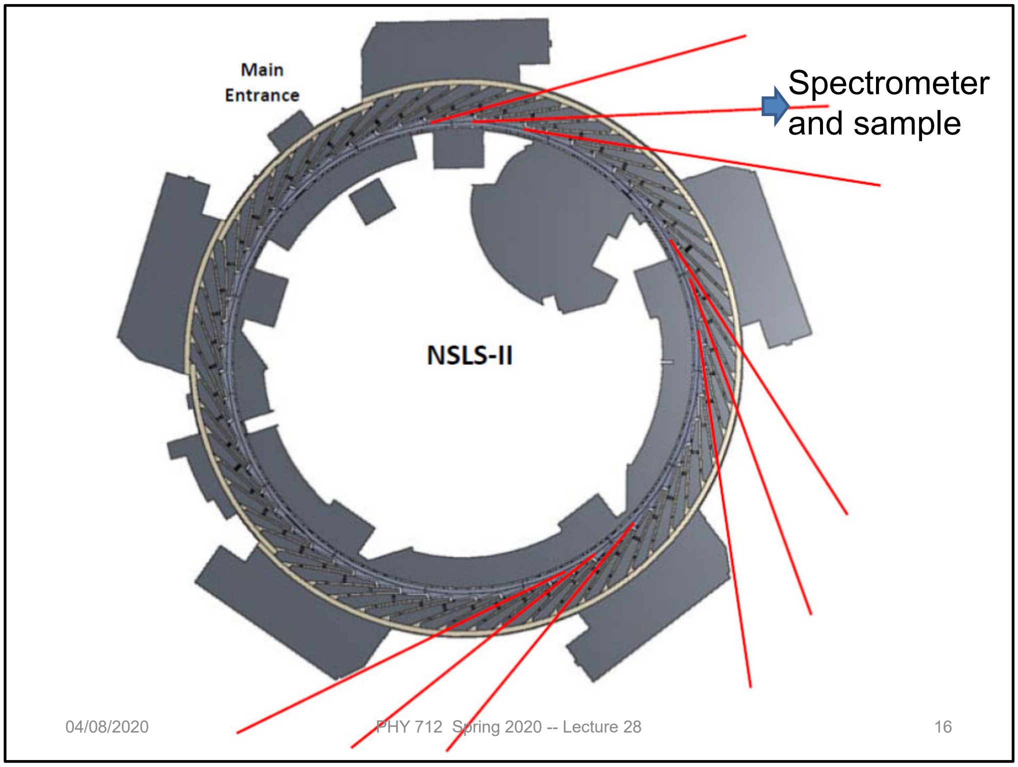
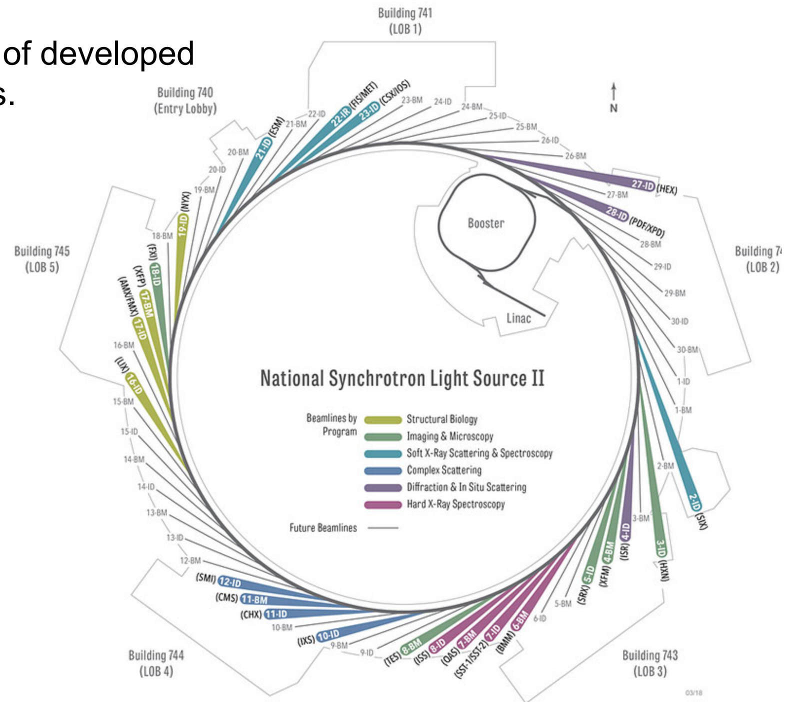


Diagram showing the positioning of the experimental ports in red where the radiation from the circulating electrons is used.

Overview of developed beamlines.



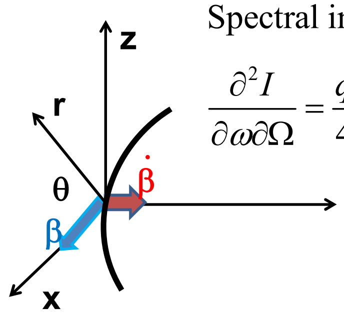
More detail with indications of particular experiments.

Advanced photon source, Argonne National Laboratory

https://www.aps.anl.gov/

Arial photo of the facility of Argonne National Laboratory

Spectral intensity relationship:



$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\;\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]\right|^2$$

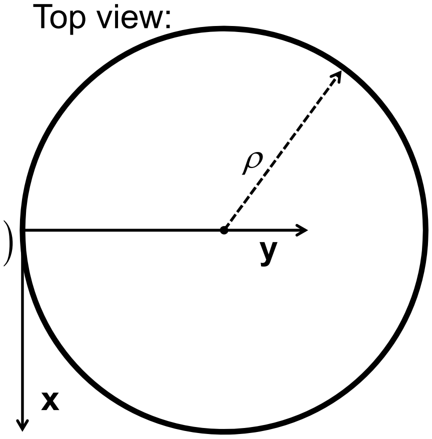
$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Top view:



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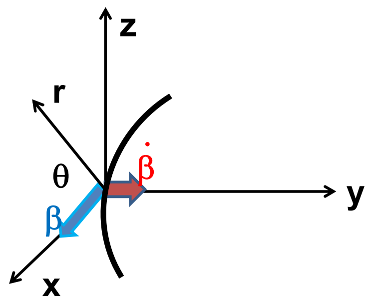
Setup of the coordinate system to analyze the particle trajectory. Here rho denotes the circular radius and v denotes the particle speed (assumed to be constant).



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

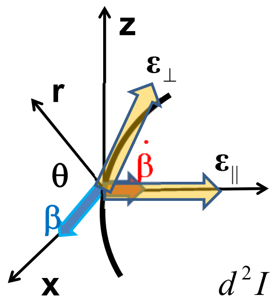
$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$

For this analysis we will assume that the radiation is detected in the x-z plane. The angle theta is defined with respect to the x axis. Two polarization vectors denoted with epsilonparallel and epsilonperpendicular (both perpendicular to r) are defined.



$$\boldsymbol{\varepsilon}_\parallel = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_\parallel \sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)} dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_\parallel(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_\parallel(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

Here the two amplitudes C for parallel and perpendicular polarizations are defined in terms of the trajectory parameters.

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \frac{3c\gamma^3}{2\rho}$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

Following Jackson's derivations, we arrive at the expression given on the slide that is equivalent to Jackson's Eq. 14.79. The two terms represent the two different polarization contributions. Both terms are expressed in terms of Bessel function of the third kind of order 1/3 or 2/3.

Some details:

Modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty dx \cos\left[\tfrac{3}{2}\xi\left(x + \tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi) = \sqrt{3}\int_0^\infty dx\, x \sin\left[\tfrac{3}{2}\xi\left(x + \tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right) = \omega\left(t_r - \frac{\rho}{c}\cos\theta\sin\left(vt_r/\rho\right)\right)$$

In the limit of $t_r \approx 0, \quad \theta \approx 0, \quad v \approx c\left(1 - \dfrac{1}{2\gamma^2}\right)$

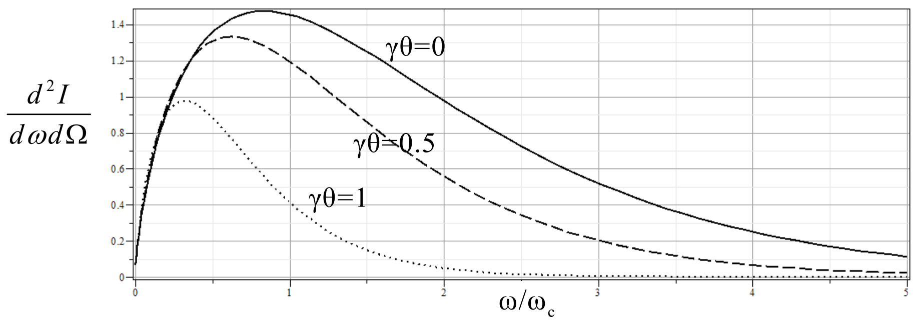
$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right) \approx \frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right) + \frac{\omega c^2 t_r^{\,3}}{6\rho^2} = \frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)$$

where $\xi = \dfrac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$ and $x = \dfrac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

Here are some of the details. It is assumed that the integrands are dominated by contributions for tr near 0, theta near 0 and speeds v close to the speed of light.

$$\frac{d^2I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:



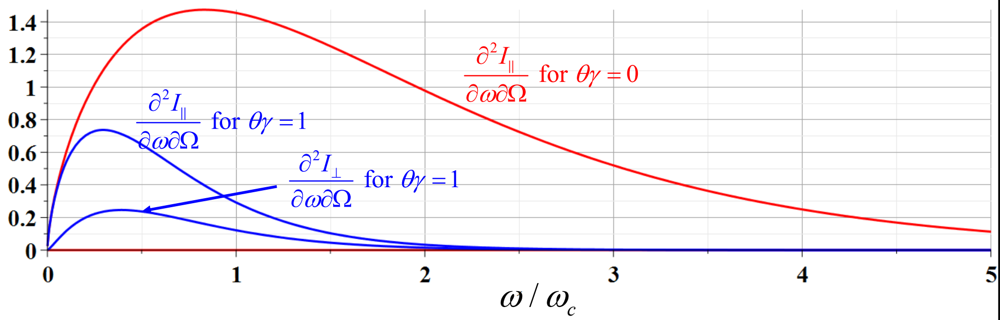
From the expression, we can plot the intensity as a function of the scaled frequency scaled by the critical frequency defined on the previous slide for various angles theta.

More details

$$\frac{d^2 I}{d\omega d\Omega} = \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$



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More plots. The fact that the frequency scales with the critical frequency means that by designing facilities with different radii and gamma factors, the spectral range can be controlled.

The above analysis applies to a class of man-made facilities dedicated to producing intense radiation in the continuous spectrum. For more specific information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.

The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of v/ρ. For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

$$\mathrm{e}^{-ix\sin u} = \sum_{m=-\infty}^{\infty} J_m(x)\mathrm{e}^{-imu} \quad \text{Here } J_m(x) \text{ is a Bessel function of integer order } m.$$

$$\text{In our case } x = \frac{\omega\rho}{c}\cos\theta \text{ and } u = \frac{vt}{\rho}.$$

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho)\mathrm{e}^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty} dt\mathrm{e}^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}).$$

Now consider another geometry for synchrotron radiation from charge particles in outer space moving in circular orbits due to magnetic fields for example. The equations are still true. We can analyze those equations using the given identity involving a series of Bessel functions.

Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}).$$

$$\Rightarrow C_{\parallel}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J_m^{'}\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}),$$

where $J_m^{'}(x) \equiv \frac{dJ_m(x)}{dx}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= 2\pi \frac{\tan\theta}{v/c} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Some details are listed here.

Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over m includes both negative and positive values. However, only the positive values of ω and therefore positive values of m are of interest. Using the identity: $J_{-m}(x) = (-1)^m J_m(x)$, the result becomes:

$$\frac{d^2 I}{d\omega d\Omega} =$$

$$\frac{q^2\omega^2\beta^2}{c}\sum_{m=0}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J'_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}.$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text.

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On the Classical Radiation of Accelerated Electrons

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This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

EARLY in 1945, much attention was focused on the design of accelerators for the production of very high energy electrons and other charged particles.[1] In connection with this activity, the author investigated in some detail the limitations to the

is instantaneously at rest is

$$P=\frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\mathbf{v}}{dt}\right)^2=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2. \quad \text{(I.1)}$$

This famous paper by Julian Schwinger discusses some of the details presented on these slides and in Jackson's textbook.

Question –

How do you imagine synchrotron radiation affecting you?

1. I can live without it
2. Good to know, but no direct effects
3. Perhaps I will take my experiment to one of the light source facilities
4. Perhaps I will model some experiments from one of the light source facilities
5. Perhaps I will be inspired to write a science fiction story about it
6. Other?