PHY 712 Electrodynamics
12-12:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Extra notes for Lecture 24:

1. **Brief comment on pure time harmonic radiation from Chap. 9 and HW 18**
2. **Start reading Chap. 11**
   - **A. Equations in cgs (Gaussian) units**
   - **B. Special theory of relativity**
   - **C. Lorentz transformation relations**

In this lecture we will jump to Chapter 11 of Jackson and the special theory of relativity. Before we do, there is one point from radiation theory that I would like to clarify.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources	#19	03/30/2020
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity	#20	04/03/2020
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity		
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles		
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

Here is the undated schedule. Notice that homework #20 is due on Friday this time.

Online colloquium scheduled for Wednesday, April 1, 2020 --
https://www.physics.wfu.edu/events/colloquium-microstructure-control-in-organic-and-hybrid-semiconductors-and-its-impact-on-device-performance

Online Colloquium: “Microstructure Control in Organic and Hybrid Semiconductors and its Impact on Device Performance “

Public talk for Ph. D. defense

Mr. Andrew Zeidell, Graduate Student

Mentor: Professor Oana Jurchescu

Department of Physics

Wake Forest University

Wednesday, April 1, 2020 at 3:00 PM

Video conference link: (available starting at 2:50 PM)
https://wakeforest-university.zoom.us/j/534312421

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Calling to your attention the colloquium for Wednesday this week. Andrew Zeidell is completing his Ph. D. thesis under the mentorship of Professor Jurchescu. This will be the public talk portion of this thesis defense. Hopefully when you are in Andrew’s shoes, you will be able to give your public talk in person. This is the last scheduled colloquium for the semester. I have sent you email about the remaining requirements for successfully completing PHY 601.

Your questions –

From Trever –

1. On slide 17, we see beta show up as a vector. Is the direction of the beta vector simply the direction in which the prime frame is moving?
2. I'm not quite sure what the last slide means by "repeated indices". Could you please explain what that refers to?

From Surya –

1. The electric and magnetic field components are expressed in terms of antisymmetric field-strength(both contravariant and covariant) tensors. Also, many other physical quantities are expressed in terms of tensor. In what situations can can we apply tensor to a given physical quantity and how?

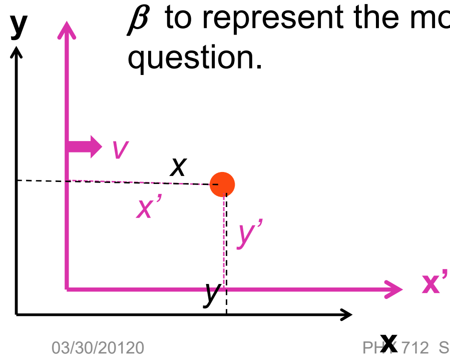
From Laxman –

1. Slide 8: Can you explain how you get the expression for r.r' there? Also the integral that follows.
2. Slide 16: Doppler effect (expression of w'/c).
3. Slide 23: how do we get the expression for transformation of acceleration.

Some answers:

Question: On slide 17, we see beta show up as a vector. Is the direction of the beta vector simply the direction in which the prime frame is moving?

Comment: For most of the lecture we have used the diagram below, keeping the relative velocity $\beta = v/c$ along the x-axis. There is nothing special about the x-axis, so we use $\boldsymbol{\beta}$ to represent the more general case as stated in the question.



Question: I'm not quite sure what the last slide means by "repeated indices". Could you please explain what that refers to?

Comment:

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

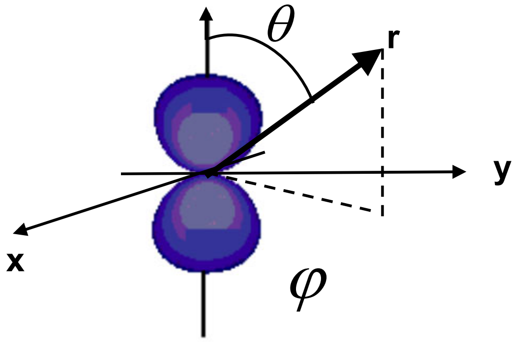
Here, the notation varies among the textbooks.
In general, it is convenient to use the matrix multiplication conventions to multiply our 4×4 matrices and 4 vectors

For example: $\mathcal{L}_v^{\alpha\beta} x'^\beta = \sum_{\beta=1}^{4} \mathcal{L}_v^{\alpha\beta} x'^\beta = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$

Question: The electric and magnetic field components are expressed in terms of antisymmetric field-strength(both contravariant and covariant) tensors. Also, many other physical quantities are expressed in terms of tensor. In what situations can we apply tensor to a given physical quantity and how?

Comment: In this lecture, we are explicitly dealing with 4 vectors and the Lorentz transformation properties work well for those 4 vectors. We will discuss the tensors next time when we consider the Maxwell's equations and we will see why and how the electric and magnetic fields have these tensor properties. Thanks for anticipating the next step which we will discuss on Wednesday.

Question: Slide 8: Can you explain how you get the expression for r.r' there? Also the integral that follows.



Note that cartesian unit vectors are fixed in space and are often very convenient for analysis.

Comment:

$$\mathbf{r} = r\left(\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\cos\phi + \hat{\mathbf{z}}\cos\theta\right)$$

$$\mathbf{r}' = r'\left(\hat{\mathbf{x}}\sin\theta'\cos\phi' + \hat{\mathbf{y}}\sin\theta'\cos\phi' + \hat{\mathbf{z}}\cos\theta'\right)$$

$$\hat{\mathbf{r}}\cdot\mathbf{r}' = r'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right)$$

More details to follow --

Slides from original lecture – with a few additions

Brief comment on pure time harmonic radiation from Chap. 9 and HW 18

From Lecture 23 --
Exact equations:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

At the end of the last lecture in the discussion of the various exact and approximate schemes for calculating radiation for pure time harmonic sources, there were perhaps some lingering questions which I would like to try to clarify using the example of HW 18.

Still exact

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Within this exact expansion we can sometimes make use of the following Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

Note that this approach is well controlled because it uses a convergent orthogonal function expansion.

Exact expansion discussed in class and various useful expansion for small or large values of the Bessel/Hankel function arguments.

Another approximation; useful, but not as well controlled

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

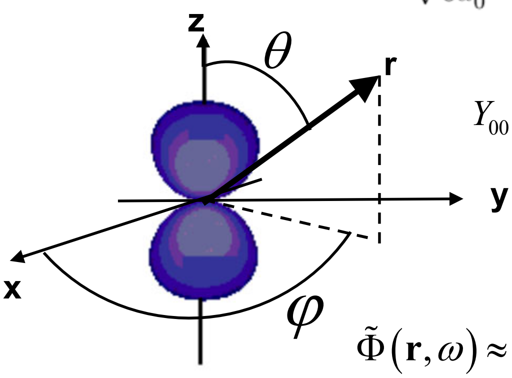
$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\; \tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\; \tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Alternatively, we introduced another approximation scheme which is often called the Born approximation.

For HW 18, you are given the following charge density distribution and asked to evaluate the scalar potential using the exact and dipole approximations. It might be interesting to see what happens when you use the "Born" approximation expression.

Charge density: $\rho(\mathbf{r}) = \frac{2e}{\sqrt{6}a_0^4}\, r \exp\left(-\frac{3r}{2a_0}\right) Y_{00}(\hat{\mathbf{r}}) Y_{10}(\hat{\mathbf{r}}) \mathrm{e}^{-i\omega_0 t}$

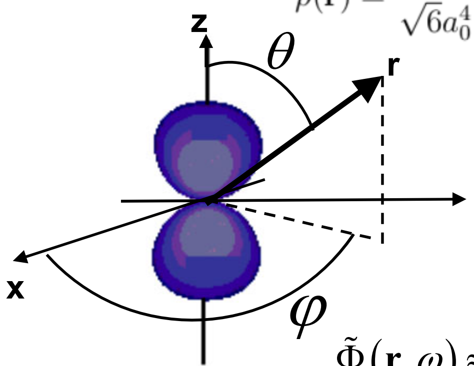


$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, \tilde{\rho}(\mathbf{r}',\omega)$$

So the question is how does the Born approximation result differ from the results you used in HW 18? This slide recalls the particular charge distribution of that HW problem.

Charge density: $\rho(\mathbf{r}) = \frac{2e}{\sqrt{6}a_0^4}\, r \exp\left(-\frac{3r}{2a_0}\right) Y_{00}(\hat{\mathbf{r}})Y_{10}(\hat{\mathbf{r}})\mathrm{e}^{-i\omega_0 t}$



$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\hat{\mathbf{r}}\cdot\mathbf{r}' = r'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right)$$

$$\int d\Omega' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\cos\theta' = -2\pi i\, j_1\left(kr'\cos\theta\right)$$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx -\frac{i}{\epsilon_0}\frac{e^{ikr}}{r}\frac{Y_{10}(\hat{\mathbf{r}})}{\sqrt{4\pi}}\frac{256\sqrt{6}ka_0e}{\left(4k^2a_0^2\cos^2\theta+9\right)^3}$$

different from HW 18

Using the diagram to evaluate the field position **r** and the source position **r’** in spherical polar coordinates, we can perform the integrals of the Born approximation. When the dust clears, we can see that the result does differ from the results obtained in HW 18.

Some more details --

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\ \tilde{\rho}(\mathbf{r}',\omega)$$

$$\int d^3r' = \int r'^2 dr'\, d\Omega' = \int r'^2 dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta' d\theta'$$

$$\int d\Omega' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \cos\theta' = -2\pi i\, j_1(kr'\cos\theta)$$ according to Maple

Units - SI vs Gaussian

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m^2/s$	$dyne$	$gm \cdot cm^2/s$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

Now – jumping to Chapter 11 of Jackson. Fortunately/unfortunately Jackson decided to use cgs Gaussian units starting in Chapter 11. Here is a table of comparison.

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$	$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

More tables of comparison of the two unit schemes.

More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

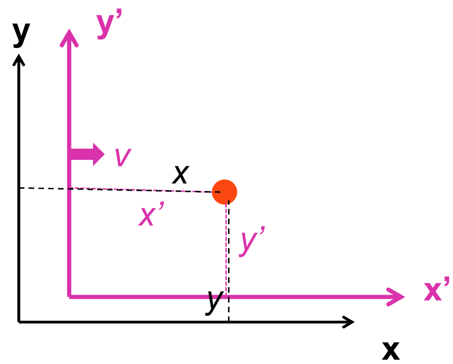
$$\epsilon \quad \Leftrightarrow \quad \epsilon / \epsilon_0$$

$$\mu \quad \Leftrightarrow \quad \mu / \mu_0$$

More relationships.

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.



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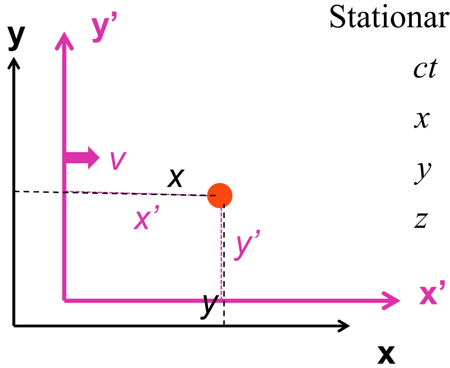
Now – jumping into the story of special relativity. These concepts were covered In Lecture 14 of PHY 742 as well. The black frame corresponds to a (stationary) frame. The purple coordinate system is moving relative to it along the x axis at a speed of v. The red dot is measured differently in the two frames of reference.

Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

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The notation with beta and gamma is defined. The question is how the four parameters, ct, x,y,z measured in the stationary reference frame are related to the corresponding four variables ct',x'y',z' measured in the moving frame and vice versa? The consensus is that the Lorentz transformation is the correct correspondence.

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

The Lorentz transformation expressed in matrix form. Also note that the four variables have an invariant.

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L}\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$

In free space:

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega'}{c}\right)^2 - k'^2 = 0$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2c^2 = E'^2 - p'^2 c^2$

Other 4 vectors that obey the Lorentz transformation --

The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

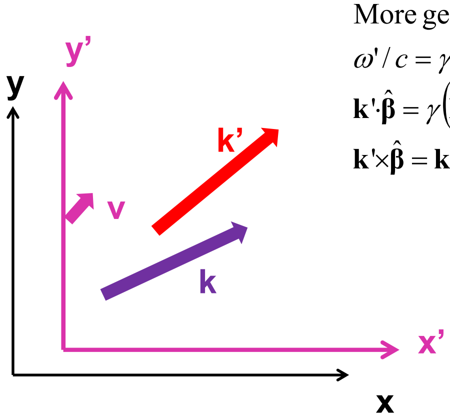
$$k'_y = k_y \qquad k'_z = k_z$$

Some details about the frequency/wavevector 4-vector and the Doppler effect for electromagnetic waves.

The Doppler Effect -- continued

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y \qquad k'_z = k_z$$



More generally:

$$\omega'/c = \gamma(\omega/c - \boldsymbol{\beta}\cdot\mathbf{k}) \equiv \gamma(\omega/c - \beta k\cos\theta)$$

$$\mathbf{k}'\cdot\hat{\boldsymbol{\beta}} = \gamma(\mathbf{k}\cdot\hat{\boldsymbol{\beta}} - \beta\omega/c) \equiv k'\cos\theta' = \gamma(k\cos\theta - \beta\omega/c)$$

$$\mathbf{k}'\times\hat{\boldsymbol{\beta}} = \mathbf{k}\times\hat{\boldsymbol{\beta}}$$

For $\theta = 0$: $(k = \omega/c)$

$$\omega' = \omega\gamma(1-\beta) \qquad \Rightarrow \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

For $\theta \neq 0$: $(k = \omega/c)$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

Doppler effect continued.

Electromagnetic Doppler Effect (θ=0)

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad \beta \approx \frac{v_{\text{source}} - v_{\text{detector}}}{c}$$

More precisely: $\beta = \dfrac{v_{\text{source}} - v_{\text{detector}}}{c\left(1 - \dfrac{v_{\text{source}}v_{\text{detector}}}{c^2}\right)}$

(Thanks to E. Carlson)

Sound Doppler Effect (θ=0)

$$\omega' = \omega\left(\frac{1 \pm v_{\text{detector}}/c_s}{1 \mp v_{\text{source}}/c_s}\right)$$

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Discussion about the Doppler effect for electromagnetic waves and sound waves.

Lorentz transformation of the velocity

Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

For an infinitesimal increment:

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Now consider the measurement of velocity in the two different reference frames.

Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt' + \beta dx')$
dx	$=$	$\gamma(dx' + \beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Define:

$$u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$$

$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx' + \beta cdt')}{\gamma(dt' + \beta dx'/c)} = \frac{u'_x + v}{1 + vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \beta dx'/c)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} = u_y$$

Evaluating the infinitesimals to determine the velocity relationships.

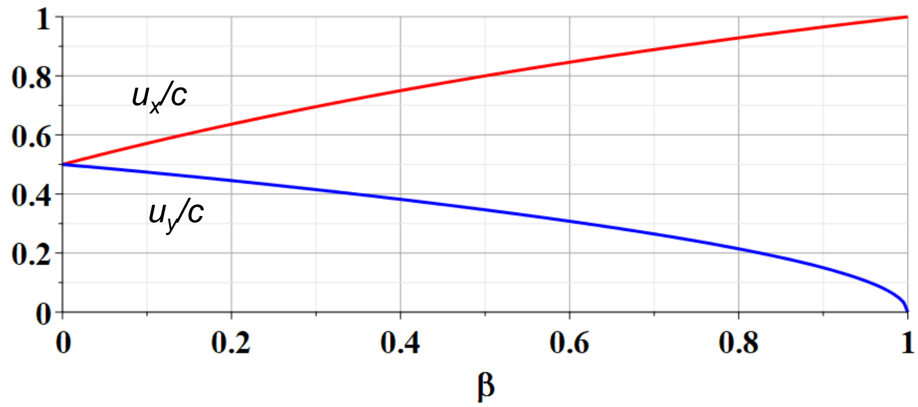
Summary of velocity relationships

$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

Example of velocity variation with β:
($u'_x/c = u'_y/c = 0.5$)



Numerical evaluation of the velocity relationship.

Extention to tranformation of acceleration

$$\mathbf{a}_{\parallel} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{\parallel}$$

$$\mathbf{a}_{\perp} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_{\perp}+\frac{\mathbf{v}}{c^2}\times\left(\mathbf{a}'\times\mathbf{u}'\right)\right)$$

It is possible to take the derivatives of the velocities to get the accelerations. The proof of these results are left for you to fill in.

Comment –
The acceleration equations are obtained by taking the infinitesimal derivative of the velocity relationships and simplifying the expressions. (See Jackson Problem 11.5.)

Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \quad u_y = \frac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \quad u_z = \frac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v\gamma_{u'}\left(1 + vu'_x/c^2\right)$

$$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'}c + \beta_v\gamma_{u'}u'_x\right)$$

$$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'}u'_x + \gamma_{u'}v\right) = \gamma_v\left(\gamma_{u'}u'_x + \beta_v\gamma_{u'}c\right)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'}u'_y \qquad \gamma_u u_z = \gamma_{u'}u'_z$$

Velocity 4-vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$

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It is apparent that the velocity 4 vector itself does not obey the Lorentz transformation. These identities show that we can construct a related 4 vector that does obey the Lorentz transformation.

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right) \quad \Rightarrow \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2$$

$$\text{where} \quad u_x = \frac{u'_x + v}{1 + vu'_x / c^2} \quad u_y = \frac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \quad u_z = \frac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2}\left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

Some identities that can be proven…

Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note}: E^2 - p^2c^2 = E'^2 - p'^2 c^2$$

When the dust clears, the related physical parameters are the energy-momentum 4 vector.

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u{}^2}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2 c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0, \quad E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1, \quad E_K = mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

In order to relate the equations to the non-relativistic treatments, we must use the same zero of energy for both. The kinetic energy of a relativistic free particle is related to the energy E-mc^2.

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check: } \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u{}^2\beta_u{}^2 + 1} = \gamma_u mc^2$$

Example: for an electron $m\mathrm{c}^2 = 0.5\,\mathrm{MeV}$

for $E = 200\,\mathrm{GeV}$

$$\gamma_u = \frac{E}{mc^2} = 4\times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u{}^2}} \approx 1 - \frac{1}{2\gamma_u{}^2} \approx 1 - 3\times 10^{-12}$$

This slide gives some numerical relationships for a highly accelerated electron.

Special theory of relativity and Maxwell's equations

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition: $$\frac{1}{c}\ \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations: $$\frac{1}{c^2}\ \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\ \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations: $$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\ \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

All of the previous equations represent relativistic mechanics. Now we want to relate the ideas to electromagnetic theory. We have said that Maxwell's equations already are consistent with the theory of relativity. But we still have some work to do in order to relate the measured fields and sources in two different reference frames. The idea is to guess the correct 4 vectors.

More 4-vectors:

$$\text{Time and position}: \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$$

$$\text{Charge and current}: \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$$

$$\text{Vector and scalar potentials}: \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$$

Here are our guesses.

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

These 4 vectors obey the Lorentz transformations. Here we use the notation that repeated indices should be summed over the 4 components. In this case beta is the summed index. Next time we will see how the E and B fields are represented in terms of the Lorentz transformations.