PHY 712 Electrodynamics
12-12:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Extra notes for Lecture 23:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

In this lecture, we will continue to focus on radiation from sources with pure harmonic time dependence with frequency omega, considering effects of superposition of multiple such sources (leading to interference) and also considering (re)radiation due to scattering of electromagnetic waves.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources	#19	03/30/2020
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity		
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity		
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles		
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

The assigned homework deals with radiation from an antenna with a slightly different configuration than covered in the textbook and in the lecture notes. Please note that we have been having one homework assigned for each lecture. This pace is designed to help you focus on the material during this time of many distractions. Please email me or request an online face to face if you encounter difficulties/questions about the homework or course content, etc.

Your questions –

From Cory:

I was looking through the scattering section, and I don't see any restrictions on the relative size of the scattering target with radius a. If this is Rayleigh scattering, shouldn't the radius be restricted to being much smaller than the wavelength range being analyzed? If the size is comparable to the wavelength range then there would presumably be Mie scattering instead, and would be wavelength independent.

From Surya:

Can you explain how can we change time averaged power distribution of radiation far from source into a pure multipole radiation with either a(lm)(E/H) given in slide 29?

Some answers –

Question: I was looking through the scattering section, and I don't see any restrictions on the relative size of the scattering target with radius a. If this is Rayleigh scattering, shouldn't the radius be restricted to being much smaller than the wavelength range being analyzed? If the size is comparable to the wavelength range then there would presumably be Mie scattering instead, and would be wavelength independent.

Comment: This is a very good point. In fact, looking at the details for the derivation of Rayleigh scattering, you see that we have made the dipole approximation which assumes that within the source $kr' \ll 1$. Mie scattering is named for Gustav Adolf Feodor Wilhelm Ludwig Mie (1868-1957) who treated light scattering more generally, allowing for general sized particles.

Some answers –

Question: Can you explain how can we change time averaged power distribution of radiation far from source into a pure multipole radiation with either a(lm)(E/H) given in slide 29?

Comment: This is probably in reference to the following equations which follow the discussion of Jackson 9.7-9.11

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_{lm}^E \mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a_{lm}^M \mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

For a pure multipole radiation with either a_{lm}^E or a_{lm}^M :

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2 \left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

Some answers continued --

Using vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

General expansion of fields using Bessel and Hankel radial functions:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Evaluating these expressions in the radiation zone ($kr >> 1$) (see page 437):

$$\mathbf{H} \approx \frac{e^{ikr-i\omega t}}{kr}\sum_{lm}\left[\left(-i\right)^{l+1}\left(a_{lm}^{E}\mathbf{X}_{lm}(\theta,\phi) + a_{lm}^{M}\left(\hat{\mathbf{r}}\times\mathbf{X}_{lm}(\theta,\phi)\right)\right)\right]$$

$$\mathbf{E} \approx Z_0\mathbf{H}\times\hat{\mathbf{r}}$$

Power can be calculted from: $\frac{dP}{d\Omega} = \frac{r^2}{2}\Re\left|\left(\hat{\mathbf{r}}\cdot\left(\mathbf{E}\times\mathbf{H}^*\right)\right)\right|^2$

Some questions – continued
Question: We have introduced a lot of different equations for time harmonic radiation, how do we know which/when to use?

Exact equations --

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Still exact

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Within this exact expansion we can sometimes make use of the following Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

Note that this approach is well controlled because it uses a convergent orthogonal function expansion.

Another approximation; useful, but not as well controlled

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> r'$: $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, \tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, \tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Slides from original lecture –

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

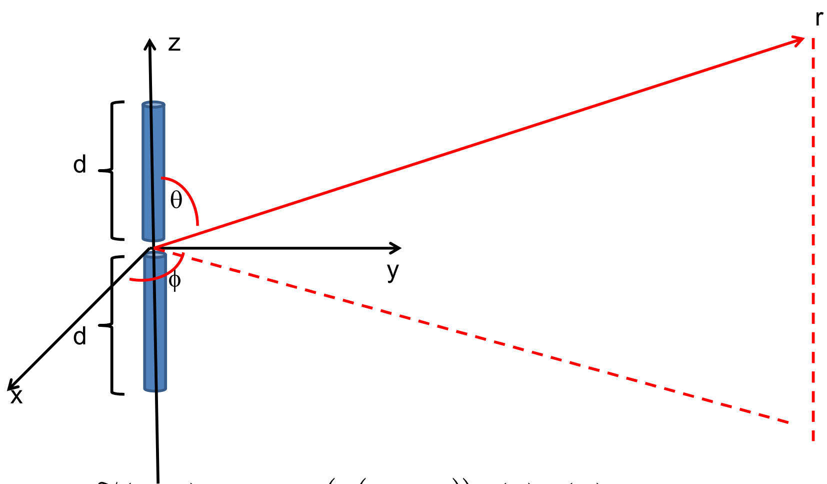
For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Review of equations that we have been using for the time Fourier transforms of the scalar and vector potentials due to their corresponding Fourier transforms of the charge and current densities.

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

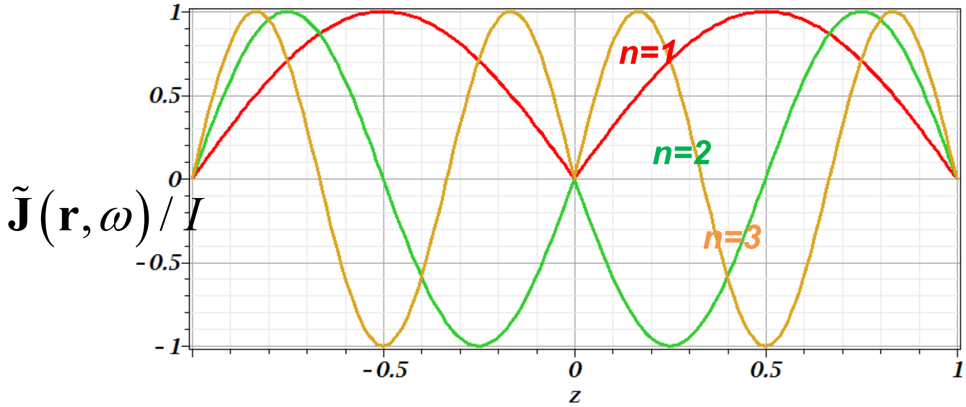
$$k \equiv \frac{\omega}{c}$$

Specifically, consider an antenna. For convenience, we are using a slightly different notation from the previous lecture as noted at the top of the slide.

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



The plot indicates how the current varies along the z axis of the antenna for the center-fed configuration.

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $\quad \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I\int_{-d}^{d} dz' \; e^{-ikz'\cos\theta} \; \sin\left(k\left(d-|z'|\right)\right)$$

Evaluation of the vector potential far from the antenna.

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I\int_{-d}^{d} dz\, e^{-ikz\cos\theta}\ \sin(k(d-|z|))$$

$$= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{kr} 2I\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin^2\theta}\right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c}{2\mu_0} r^2\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}}\cdot\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

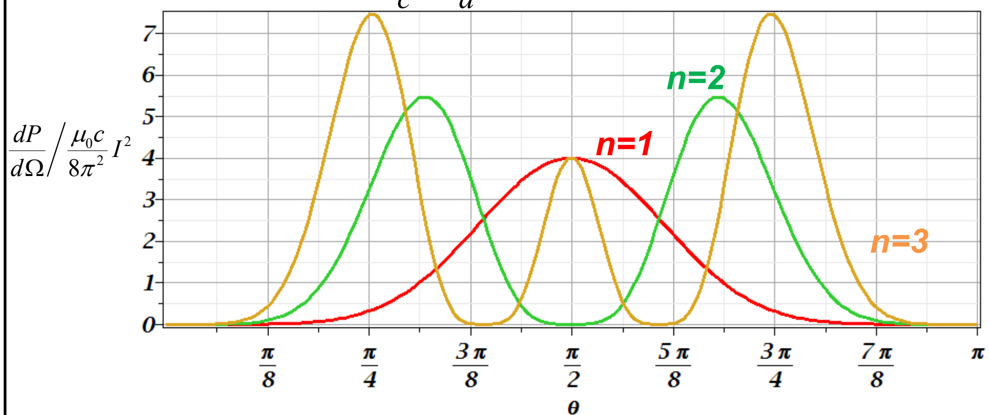
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

Some details for evaluating the power per unit solid angle.

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



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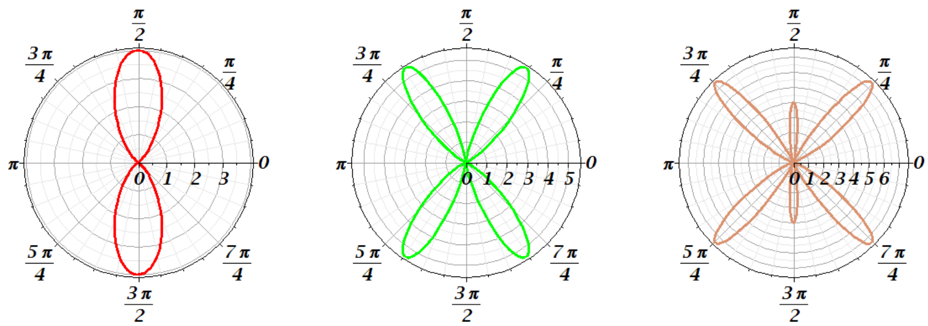
Plot of the power distribution as a function of angle for this case.

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

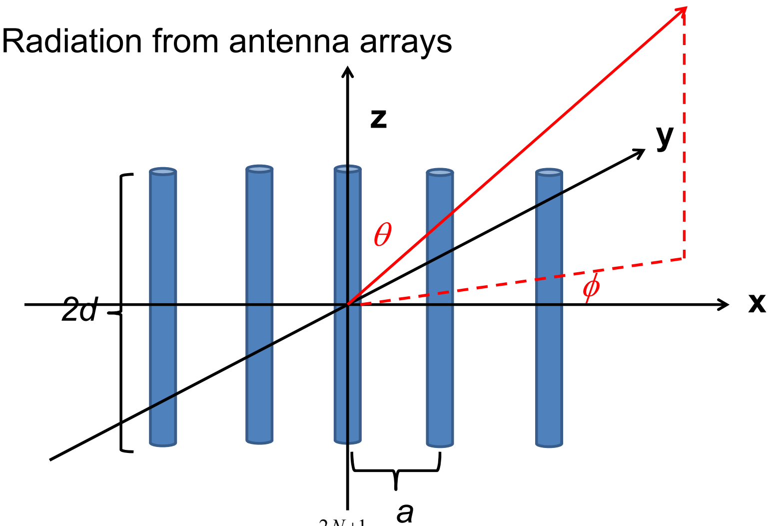
For $kd = n\pi$:



n=1 ***n=2*** ***n=3***

Polar plots of the power distribution.

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right) \sum_{j=1}^{2N+1} \delta\left(x-(N+1-j)a\right)\delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3....$$

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Now consider the case of several antennas, in this case each antenna is oriented along the z-axis and 2N+1 of them are arranged in a line along the x-axis.

Radiation from antenna arrays -- continued

Vector potential from array source:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\widetilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\widetilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\widetilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin(k(d-|z|))\sum_{j=1}^{2N+1}\delta(x-(N+1-j)a)\delta(y)\quad \text{for } -d\le z\le d$$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\ \sin(k(d-|z|))$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Analyzing the same equations as before, keeping the leading terms for the limit that *kr*→infinity. Here we see that the x-axis dependence involves evaluating a geometric series which can be done analytically as shown.

Radiation from antenna arrays -- continued

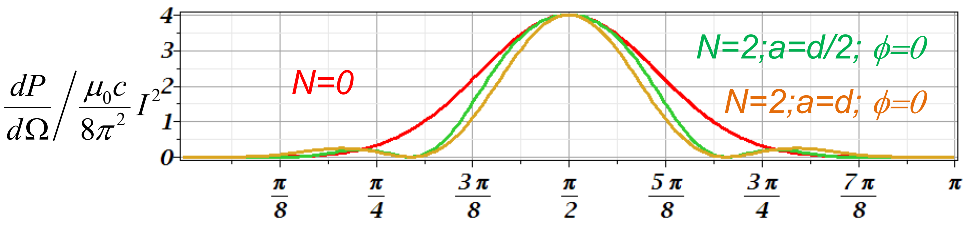
In the radiation zone:

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega)=\nabla\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)\approx ik\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega)\approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

$$\frac{dP}{d\Omega}=\frac{1}{2\mu_0}r^2\hat{\mathbf{r}}\cdot\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)=\frac{k^2cr^2}{2\mu_0}\left(\left|\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2-\left|\hat{\mathbf{r}}\cdot\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

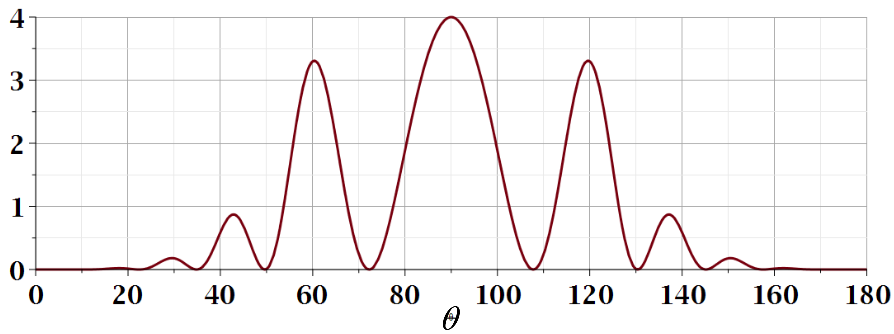
$$\frac{dP}{d\Omega}=\frac{\mu_0 c}{8\pi^2}I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2\left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



Carrying out the integrations and simplifying the expressions, we get the results. The plots here refer to phi=0, which corresponds to the observation of the radiation along the x-axis.

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)} \right]^2$$

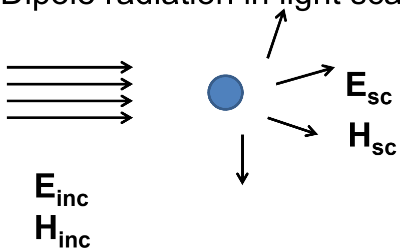
Example for $\phi = 0, N = 10, \quad kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Plot of the power for another case. Obviously, there is a lot of variety with antenna arrays which are used extensively for communications and other technologies.

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left(\left(\hat{\mathbf{r}} \times \mathbf{p} \right) \times \hat{\mathbf{r}} \right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

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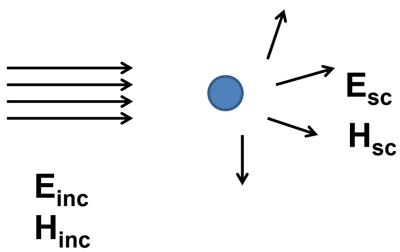
Now consider a different radiation source – that is re-radiation from matter interacting with light (such as sunlight). Here we will simplify the analysis and assume that the matter is in the form of uniform sphere. This topic is covered in Chapter 10 of Jackson.

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

Scattering cross section :

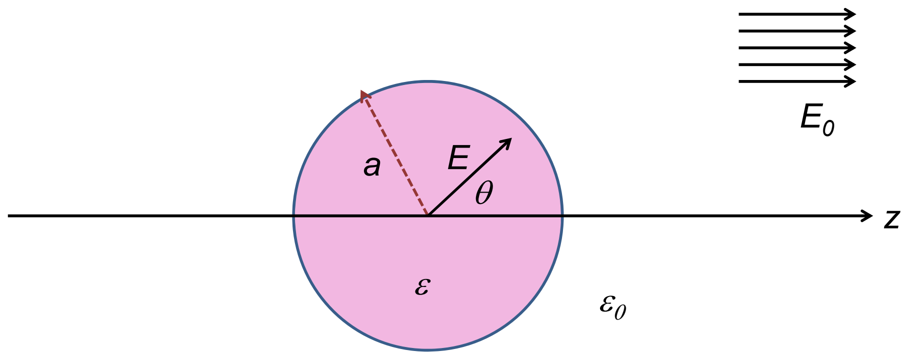
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{sc} \rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{inc} \rangle_{avg}}$$

$$= \frac{r^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{p}\right|^2$$

We will assume that the incident light is in the form of an ideal plane wave, and analyze the re-radiated light as a spherical wave far from the particle itself. The unit vectors epsilon_0 and epsilon reference the incident polarization of the light and the scattered polarization direction of the light, respectively. The cross section is defined as the scattered power per unit incident power.

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

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Analyzing the source of re-radiation, we need to recall how a spherical dielectric of radius a interacts with a constant electric field. We can use the results we obtained in Chapter 4 when we considered the situation as an electrostatic boundary value problem. Here the z direction is the direction of the incident electric field, not the wave vector direction.

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty$ $\Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

These are the results from the electrostatic case discussed previously.

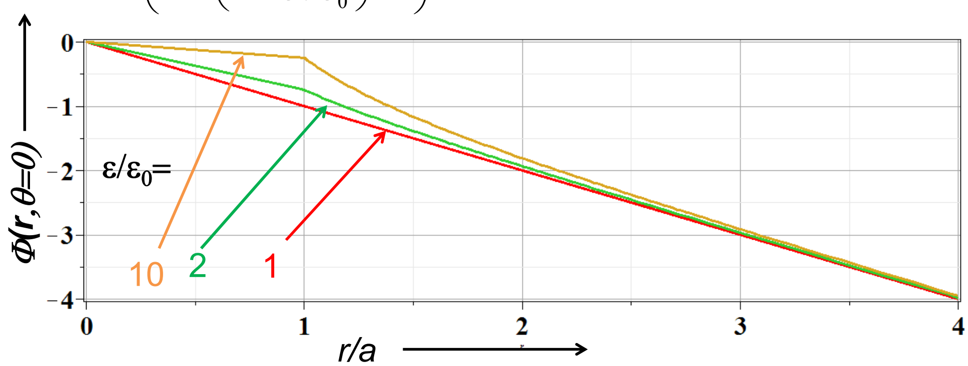
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0\cos\theta$$

Induced dipole moment:

$$\mathbf{p} = 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Continued results obtained previously for the electrostatic problem.

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

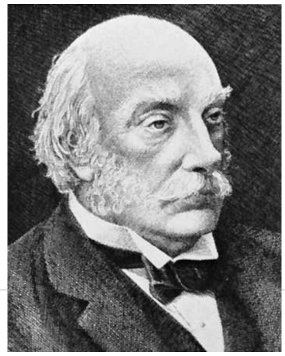
Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

Jumping back to the scattering problem, assuming that the same mathematics can be translated to this case -- Here we have used bold epsilon to reference the polarization directions. These directions are always perpendicular to the light propagation directions. The not bold epsilons indicate the permittivity functions which are functions of the harmonic frequency of the light involved. The final result was derived by Lord Raleigh.

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh



WRITTEN BY: R. Bruce Lindsay
See Article History

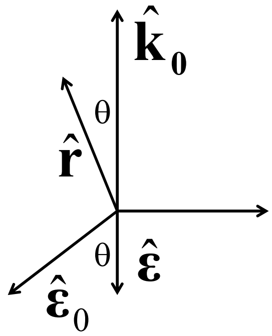
Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Some information about Lord Rayleigh on the web.

Scattering by dielectric sphere with permittivity ε and radius *a*:

For $\mathbf{E}_{inc}$ polarized in scattering plane:

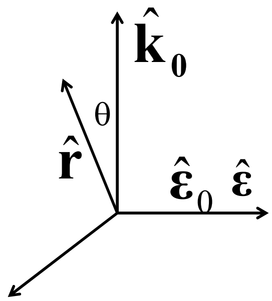


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) == k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

In this analysis, we consider the case where the incident wavevector (along the vertical axis) and the polarization direction (epsilon0) are in the same plane as the observed scattered light (direction of \hat{r}). In this case , the dot product of the incident and scattered polarizations give a factor of cos(theta) as show.

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{\mathbf{inc}}$ polarized perpendicular to scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

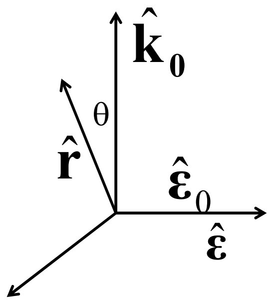
$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

Assuming both polarizations are equally likely, average cross section is given by :

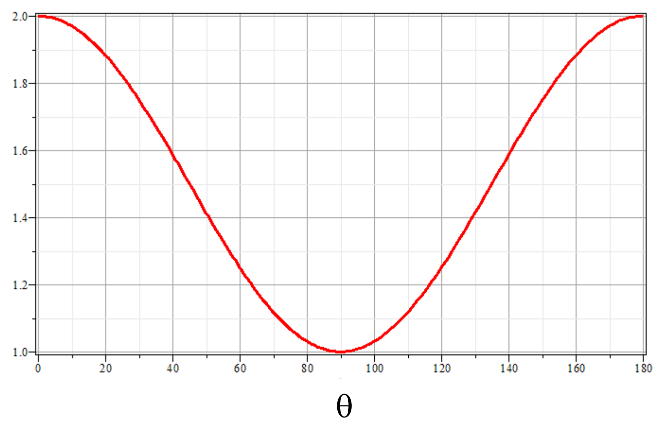
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

In this case, the incident wavevector (along the vertical axis) and the observed scattered light (direction of \hat{r}) are as before and again define the scattering plane. However, the polarization direction of incident light (epsilon0) and the polarization direction of the scattered light (epsilon) are both perpendicular to the scattering plane and thus are parallel to each other, given 1 for their dot product. The last result indicates the cross section of the total scattered light assuming both polarizations are equally likely.

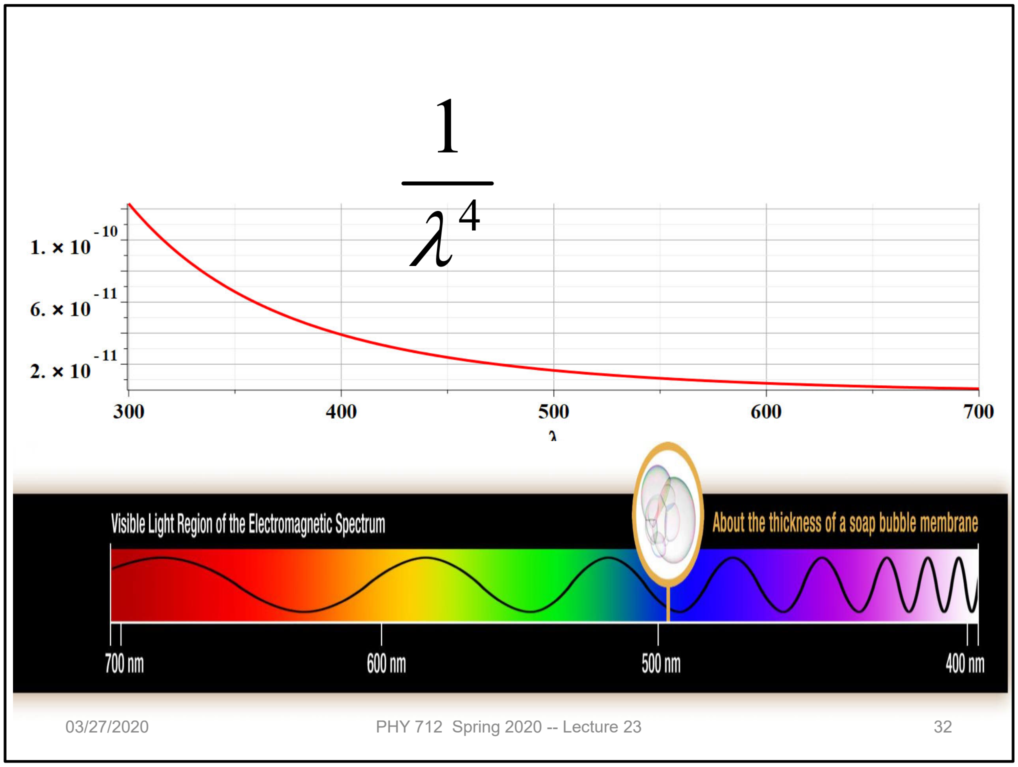
Scattering by dielectric sphere with permittivity ε and radius *a*:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4 a^6}{2}\left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left(\cos^2\theta + 1\right)$$



The plot shows the angular dependence of the scattered light as a function of the angle theta.



In addition to the angular dependence of the scattered light, Raleigh scattering depends of the wavevector as k^4 which has the corresponding wavelength dependence indicated on this slide.   The figure from the web shows the variation of wavelength for visible light. The analysis of Raleigh scattering thus tells us why the sky at mid day is blue and why it tends to be red at sun rise and sunset.

Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of $\mathbf{E}$ and $\mathbf{H}$ fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

In the next few slides, we go over material presented in Section 9.7 of your textbook. I have personally never used this formalism, but recognize it as a powerful tool for analyzing fields from localized sources in terms of the fields themselves rather than using scalar and vector and scalar potentials. Please review this material as time permits.

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r}\cdot\mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r}\cdot\mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L}\cdot\mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r}\cdot\mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r}\cdot\mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L}\cdot\mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm} (-i)^{l+1}\left[a^E_{lm}\mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a^M_{lm}\mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

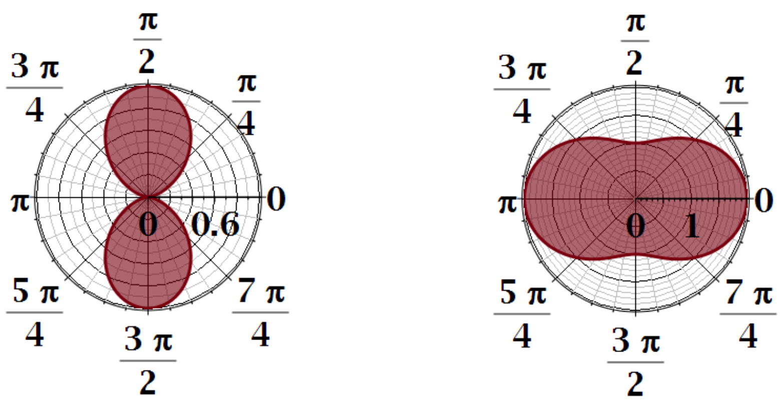
For a pure multipole radiation with either a^E_{lm} or a^M_{lm} :

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l=1$

$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2 = \frac{3}{8\pi}\sin^2\theta$ $\left|\mathbf{X}_{11}(\theta,\phi)\right|^2 = \left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)$



For example: $l = 2$

$$|\mathbf{X}_{20}(\theta,\phi)|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad |\mathbf{X}_{21}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad |\mathbf{X}_{22}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$$