

PHY 712 Electrodynamics
12-12:50 AM MWF via video link:
<https://wakeforest-university.zoom.us/my/natalie.holzwarth>

Plan for Lecture 22:

Continue reading Chap. 9 & 10

**A. Electromagnetic waves due to
specific sources**

B. Dipole radiation examples

C. Scattered radiation

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Welcome to the second annotated lecture for PHY 712. This lecture continues our discussion of electromagnetic waves produced by sources with reference to Jackson's textbook, Chapter 9.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources		
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity		
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity		
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles		
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	<i>Good Friday</i>		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

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The new assignment #18 builds on the analysis from the previous homework (#17)..

Online colloquium on Wednesday (today) –

<https://www.physics.wfu.edu/events/rick-matthews-41-years/>

Online Colloquium: “41 Years of Teaching and
Technology”

Dr. Rick Matthews
Professor of Physics and
Director of Academic and Instructional Technology
Wake Forest University

Wednesday, March 25, 2020 at 3:00 PM

Video conference link: [https://wakeforest-university.zoom.us](https://wakeforest-university.zoom.us/j/zoom-link)
[/my/matthews.rick](https://wakeforest-university.zoom.us/j/zoom-link)

Note: this is an online Zoom presentation. If you have not used Zoom recently, click on the above link to join about ten minutes early to be sure the necessary software installs itself.

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Remember to link to the first online colloquium from Professor Rick Matthews at 3 PM.

Review:

Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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First we need to review the equations and results from last time. Here we are working with Maxwell's equation for the E and B fields responding to sources as characterized by their charge and current densities. It is assume that polarization and magnetization is zero.

Review:

Formulation of Maxwell's equations in terms of vector and scalar potentials:

$$\text{Lorenz gauge form -- require: } \nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Note that the Lorenz gauge is consistent with the

$$\text{source continuity condition: } \frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0$$

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It is convenient to analyze the equations in terms of the scalar or vector potentials using the Lorenz gauge.

Review:

Electromagnetic waves from time harmonic sources

$$\text{Charge density: } \rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega)e^{-i\omega t})$$

$$\text{Current density: } \mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega)e^{-i\omega t})$$

$$\Rightarrow \text{Scalar potential: } \Phi(\mathbf{r}, t) = \Re(\tilde{\Phi}(\mathbf{r}, \omega)e^{-i\omega t})$$

$$\Rightarrow \text{Vector potential: } \mathbf{A}(\mathbf{r}, t) = \Re(\tilde{\mathbf{A}}(\mathbf{r}, \omega)e^{-i\omega t})$$

For $k \equiv \frac{\omega}{c}$:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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We will consider sources with pure harmonic time dependence with the notion that any real source can be represented by a linear combination of these pure sources via a Fourier transform.

Review:

Electromagnetic waves from time harmonic sources –
continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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This identity was introduced last time, allowing us to represent the scalar and vector potentials as a spherical harmonic expansion.

Review:

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x}$$

$$h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$$

$$h_1(x) = -\left(1 + \frac{i}{x}\right) \frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin(x) - \frac{3 \cos(x)}{x^2}$$

$$h_2(x) = i \left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x \ll 1 \quad \Rightarrow \quad j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x \gg 1 \quad \Rightarrow \quad h_l(x) \approx (-i)^{l+1} \frac{e^{ix}}{x}$$

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These results are given in Jackson's text.

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos\theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$\tilde{\mathbf{A}}(r, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr} \int_0^r r' dr' e^{-r'/R} \sin(kr') + \frac{\sin(kr)}{kr} \int_r^\infty r' dr' e^{-r'/R + ikr'} \right)$$

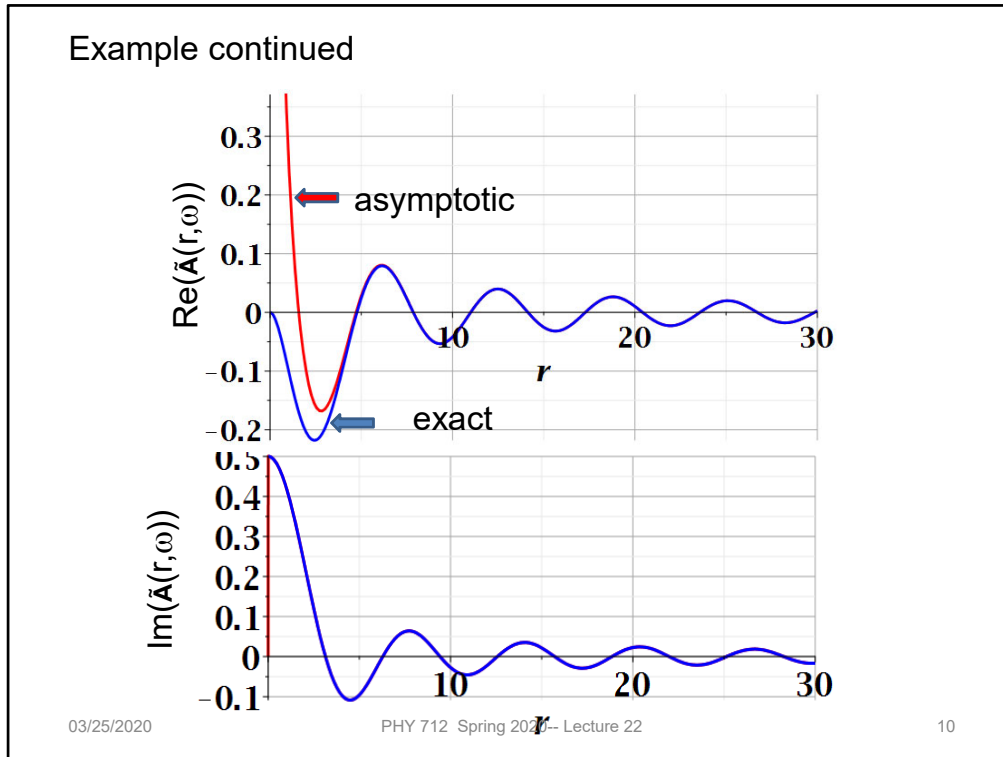
$$\underset{r \gg R}{\approx} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{kr} \frac{2Rk^2 R^2}{(k^2 R^2 + 1)^2}$$

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An example of using the spherical harmonic expansion to analyze “exact” expressions for the scalar and vector potentials.



Comparing the values of the vector potential calculated using the asymptotic expansion ($r \rightarrow \infty$) with the exact evaluation. You see that the difference occurs only within the source extent.

Review:

Electromagnetic waves from time harmonic sources –
continued:

Dipole radiation case:

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

For fields outside extent of source and $kr' \ll 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

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In the long wavelength limit, the dipole approximation is numerically close to the physical situation.

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r}, \omega) &= -\nabla\tilde{\Phi}(\mathbf{r}, \omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 \left((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{B}}(\mathbf{r}, \omega) &= \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0 c} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr} \right)\end{aligned}$$

Power radiated for $kr \gg 1$:

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega)) \\ &= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2\end{aligned}$$

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Continued analysis of vector and scalar potential fields following Jackson Section 9.2.

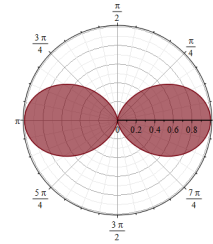
Properties of dipole radiation field for $kr \gg 1$:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) \right)$$

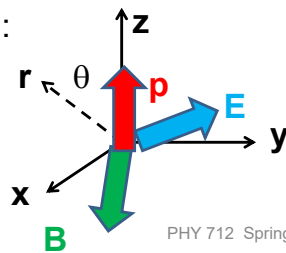
$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega))$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$



Example:



Note that vectors \mathbf{r} , \mathbf{E} , \mathbf{B} are mutually orthogonal

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Geometric properties of dipolar fields.

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For $r \gg r'$: $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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Up to now, we have worked with the exact spherical harmonic expansion, evaluating the results in certain limits. Now consider analyzing the Green's function integral directly without use of Bessel functions. You may recognize this treatment as the Born approximation encountered in quantum mechanical scattering theory.

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\text{For } r \gg r': \quad |\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

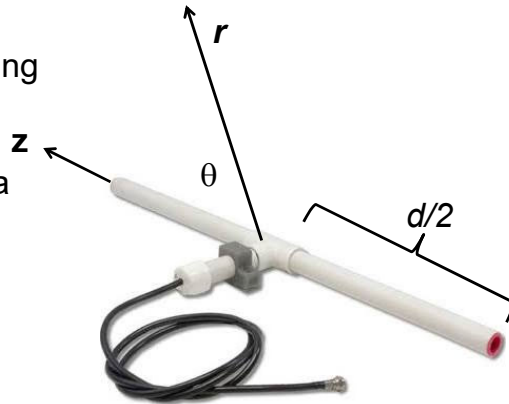
$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

→ Results equivalent to Bessel function expansion
in the limit $kr \rightarrow \infty$.

This approach is similar to the Bessel function expansion if more terms were used.

Other radiation sources using
"alternative approach"

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx$$

$$\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}', \omega) = I_0 \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\delta(y)\hat{\mathbf{z}}$$

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Up to now, we have been thinking about sources on the atomic scale. The analysis also works for macroscopic sources such as antennas. This example which follows your textbook is called a center fed antenna.

Alternative approach – linear center-fed antenna continued

$$\begin{aligned}\tilde{\mathbf{A}}(\mathbf{r}, \omega) &\approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi r} e^{ikr} \int_{-d/2}^{d/2} dz' e^{-ik \cos(\theta) z'} \sin\left(\frac{kd}{2} - k|z'|\right) \\ &= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi kr} e^{ikr} \left(\frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2 \theta} \right)\end{aligned}$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

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Analyzing the vector potential in this “Born” approximation, we obtain an analytic result for the radiation distribution.

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

$$\text{for } kd = \pi: \quad \frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$\text{for } kd = 2\pi: \quad \frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

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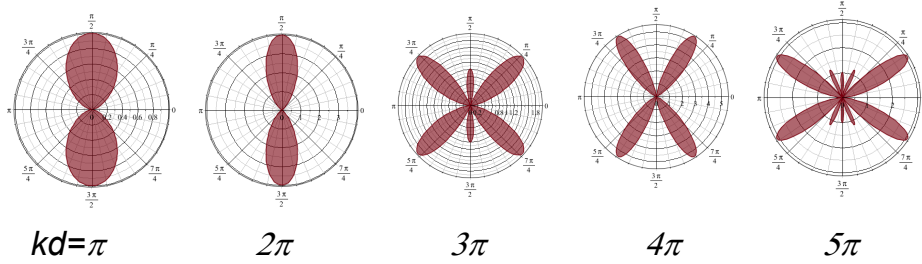
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The radiation pattern is quite sensitive to the relationship between the antenna length d and the wavelength of the radiation.

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$



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Here are some polar plots of the radiation patterns for various cases.

Another source of radiation –
Radiation due particles reacting to incident
electromagnetic waves – scattering processes
Chapter 10 in **Jackson**

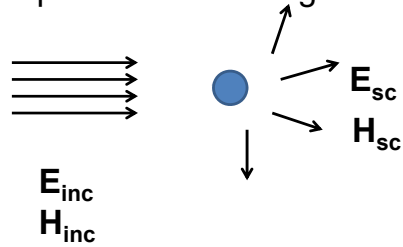
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Now consider another radiation source – scattered light. We will introduce the topic (covered in Chapter 10) today, but discuss it more thoroughly on Friday.

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}} \quad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \quad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

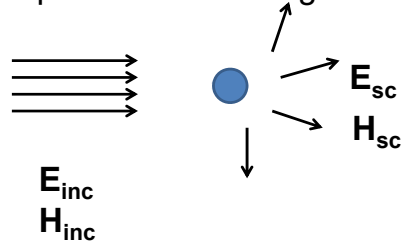
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Imagine a plane wave of light incident on a sphere. The incident light produces oscillating dipoles within the sphere which in turn produce dipole radiation.

Dipole radiation in light scattering by small (dielectric) particles



Scattering cross section :

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) &= \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{sc} \rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{inc} \rangle_{avg}} \\ &= \frac{r^2 |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\boldsymbol{\epsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{p}|^2 \end{aligned}$$

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The radiation depends on the initial polarization of the scattered polarization.

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ϵ and radius a :

$$\mathbf{p} = 4\pi a^3 \epsilon_0 \left(\frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right) \mathbf{E}_{inc} \quad \mathbf{E}_{inc} = \hat{\mathbf{\epsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section :

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) &= \frac{r^2 |\hat{\mathbf{\epsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{\epsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{\epsilon}} \cdot \mathbf{p}|^2 \\ &= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2 \end{aligned}$$

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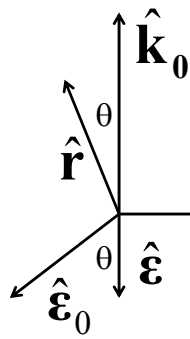
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Using results from the electrostatic polarization analysis, we can deduce the polarization amplitude.

Scattering by dielectric sphere with permittivity ϵ and radius a :

For \mathbf{E}_{inc} polarized in scattering plane:



$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) &= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}}_0|^2 \\ &= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 \cos^2 \theta \end{aligned}$$

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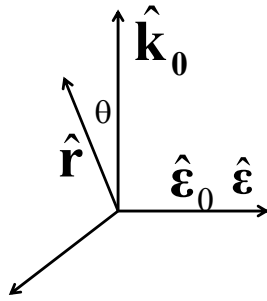
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We will continue this discussion on Friday, considering the geometrical effects.

Scattering by dielectric sphere with permittivity ϵ and radius a :

For \mathbf{E}_{inc} polarized perpendicular to scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2$$

Assuming both polarizations are equally likely, average cross section is given by :

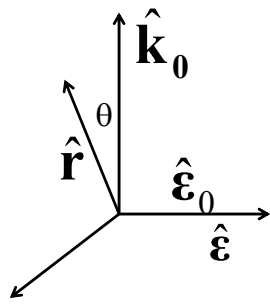
$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

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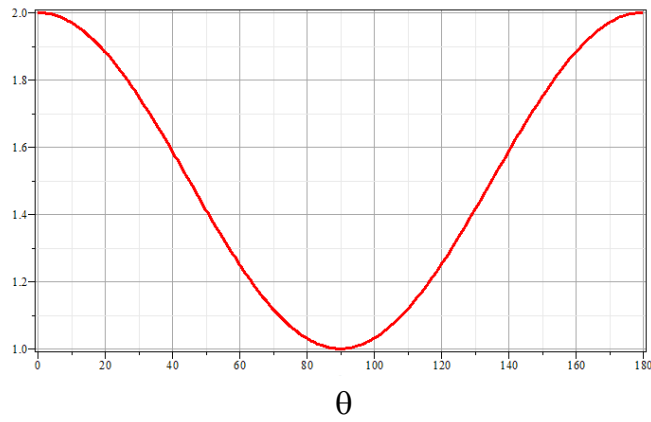
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Scattering by dielectric sphere with permittivity ϵ and radius a :



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$



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