PHY 712 Electrodynamics
12-12:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Extra notes for Lecture 22:

Continue reading Chap. 9 & 10

- A. Electromagnetic waves due to specific sources
- B. Dipole radiation examples
- C. Scattered radiation

03/25/2020 PHY 712 Spring 2020-- Lecture 22 1

Welcome to the second annotated lecture for PHY 712. This lecture continues our discussion of electromagnetic waves produced by sources with reference to Jackson's textbook, Chapter 9.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources		
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity		
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity		
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles		
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

The new assignment #18 builds on the analysis from the previous homework (#17)..

Online colloquium on Wednesday (today) –
https://www.physics.wfu.edu/events/rick-matthews-41-years/

Online Colloquium: “41 Years of Teaching and Technology”

Dr. Rick Matthews
Professor of Physics and
Director of Academic and Instructional Technology
Wake Forest University
Wednesday, March 25, 2020 at 3:00 PM
Video conference link: **https://wakeforest-university.zoom.us/my/matthews.rick**

Note: this is an online Zoom presentation. If you have not used Zoom recently, click on the above link to join about ten minutes early to be sure the necessary software installs itself.

Remember to link to the first online colloquium from Professor Rick Matthews at 3 PM.

Your questions –

From Trevor:

1. How exactly do we find rho(r,omega) if we're given rho(r,t)? On the homework I just assumed that we would use a fourier transform, but I wasn't 100% sure at the time.
2. On slide 17, could you please explain what method was used to find the time averaged power in this example? I know that there are different ways to calculate it, and I was having trouble seeing which one is used here.

From Surya:

1. In Jackson, scalar electric potential in Lorentz gauge and the long-wavelength limit is given by; $\phi(\boldsymbol{r})$=$eikr$/4$\pi\varepsilon$0r2* $\boldsymbol{n}.\boldsymbol{p}$(1−$ikr$) (exercise 9.2). Is this same as last equation of slide 11? (Perhaps 9.5?)

From Laxman:

1. Slide 17: How is the time averaged power calculated?
2. How do the diagrams in slide 18 represent time averaged powers at different angles?
3. Slide 23: Have we used this expression of scattering cross section before?

Some answers –

Question: How exactly do we find rho(r,omega) if we're given rho(r,t)? On the homework I just assumed that we would use a Fourier transform, but I wasn't 100% sure at the time.

Comment: In general, if you know $\rho(\mathbf{r},t)$, the Fourier transform in the time domain is defined:

$$\tilde{\rho}(\mathbf{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\rho(\mathbf{r},t)e^{i\omega t}$$

The inverse transform is :

$$\rho(\mathbf{r},t) = \int_{-\infty}^{\infty} d\omega\,\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}$$

Unfortunately, details vary among texts (even within texts) about factors of 2π and "forward" vs "backward" transforms. In your homework, the focus was on a single term of $\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}$.

Some answers –

Question: In Jackson, scalar electric potential in Lorentz gauge and the long-wavelength limit is given by; $\phi(\boldsymbol{r})$=$eikr$/4$\pi\varepsilon$0r2* $\boldsymbol{n}.\boldsymbol{p}$(1−$ikr$) (exercise 9.2). Is this same as last equation of slide 11? (Perhaps 9.5?)

Comment: Yes, these are the same, after correcting a typo. (Thanks for catching it. Slide 22 Lecture 21 is now corrected.)

From slide 11 after correction:

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

From Jackson problem 9.5:

$$\Phi(\mathbf{x}) = \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\mathbf{n}\cdot\mathbf{p}\left(1-ikr\right)$$

Some answers –

Question: On slide 17, could you please explain what method was used to find the time averaged power in this example? I know that there are different ways to calculate it, and I was having trouble seeing which one is used here.

Slide 17: How is the time averaged power calculated?

Comment: These details are presented in Jackson section 9.2 The following slides go over some of the equations.

Power in the dipole approximation; Section 9.2 of Jackson

Here we use our notation with $\mathbf{n} \rightarrow \hat{\mathbf{r}}$ and $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \Re \left| \left(\hat{\mathbf{r}} \cdot \left(\mathbf{E} \times \mathbf{H}^* \right) \right) \right|^2$$

Using the expressions for the dipole fields far from the source:

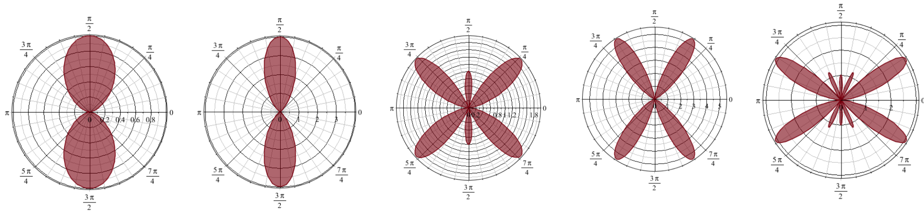
$$\mathbf{H} = \frac{ck^2}{4\pi} \left(\hat{\mathbf{r}} \times \mathbf{p} \right) \frac{e^{ikr}}{r} \qquad \mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}}$$

The power can be written $\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| \left(\left(\hat{\mathbf{r}} \times \mathbf{p} \right) \times \hat{\mathbf{r}} \right) \right|^2$

Defining the angle θ by $\mathbf{p} \cdot \hat{\mathbf{r}} = |\mathbf{p}| \cos\theta$,

$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta$ integrating over solid angles $P = \frac{c^2 Z_0}{12\pi} k^4 |\mathbf{p}|^2$

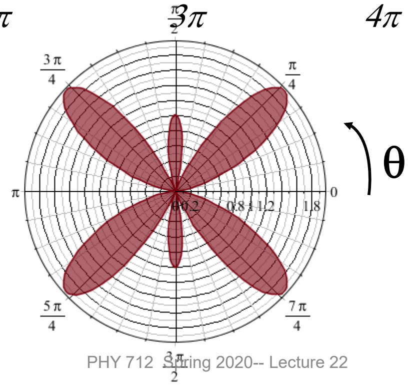
Question: 2. How do the diagrams in slide 18 represent time averaged powers at different angles?



kd=π 2π 3π 4π 5π

Polar plot:
Angle indicates values of theta

Radius indicates value scaled to 1.



Slides from original lecture --

Review:

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

03/25/2020 PHY 712 Spring 2020-- Lecture 22 11

First we need to review the equations and results from last time. Here we are working with Maxwell's equation for the E and B fields responding to sources as characterized by their charge and current densities. It is assume that polarization and magnetization is zero.

Review:

Formulation of Maxwell's equations in terms of vector and scalar potentials:

Lorenz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Note that the Lorenz gauge is consistent with the source continuity condition: $\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0$

It is convenient to analyze the equations in terms of the scalar of vector potentials using the Lorenz gauge.

Review:
Electromagnetic waves from time harmonic sources

$$\text{Charge density:} \quad \rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

$$\text{Current density:} \quad \mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

$$\Rightarrow \text{Scalar potential:} \quad \Phi(\mathbf{r},t) = \Re\left(\tilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

$$\Rightarrow \text{Vector potential:} \quad \mathbf{A}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{A}}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

$$\text{For } k \equiv \frac{\omega}{c}:$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

We will consider sources with pure harmonic time dependence with the notion that any real source can be represented by a linear combination of these pure sources via a Fourier transform.

Review:

Electromagnetic waves from time harmonic sources – continued:

Useful expansion:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

This identity was introduced last time, allowing us to represent the scalar and vector potentials as a spherical harmonic expansion.

Review:

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

These results are given in Jackson's text.

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

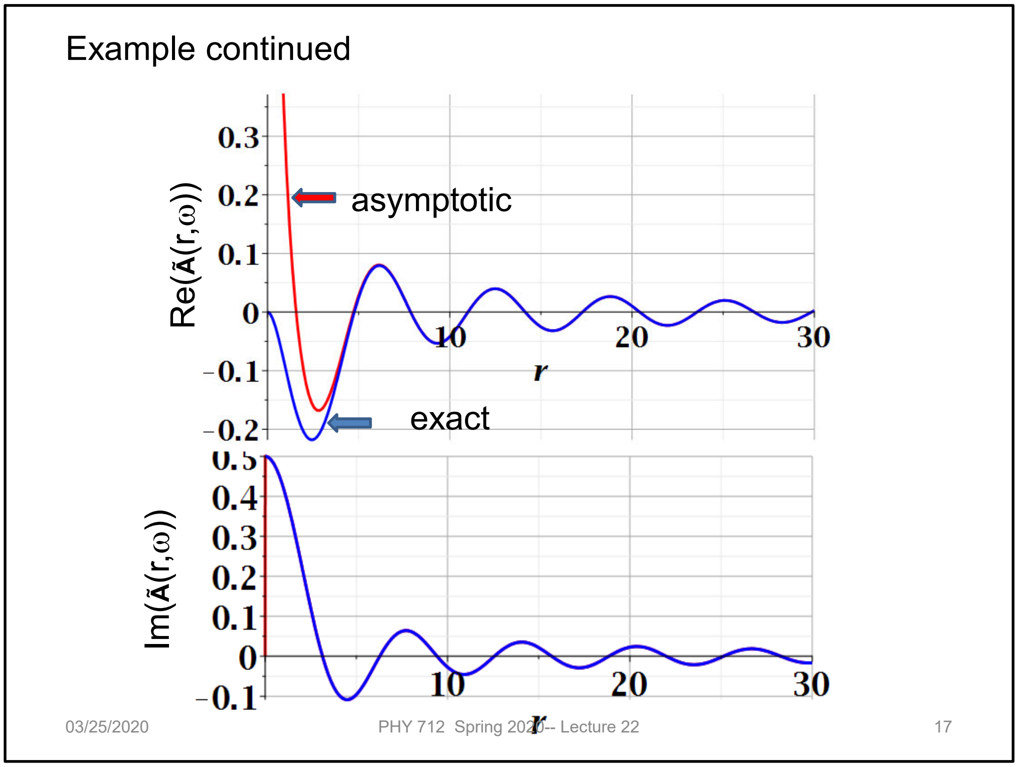
$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \,(ik\mu_0)\int_0^\infty r'^2\, dr'\, e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R}\cos\theta \int_0^\infty r'^2\, dr'\, e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$\tilde{\mathbf{A}}(r,\omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr}\int_0^r r'\, dr' e^{-r'/R}\sin(kr') + \frac{\sin(kr)}{kr}\int_r^\infty r'\, dr' e^{-r'/R+ikr'} \right)$$

$$\underset{r \gg R}{\approx} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{kr}\frac{2Rk^2R^2}{\left(k^2R^2+1\right)^2}$$

An example of using the spherical harmonic expansion to analyze “exact” expressions for the scalar and vector potentials.



Comparing the values of the vector potential calculated using the asymptotic expansion (r→infinity) with the exact evaluation. You see that the difference occurs only within the source extent.

Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r\ \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

For fields outside extent of source and $kr' << 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

In the long wavelength limit, the dipole approximation is numerically close to the physical situation.

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right) - \mathbf{p}(\omega)}{r^2}\right)(1 - ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1 - \frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Continued analysis of vector and scalar potential fields following Jackson Section 9.2.

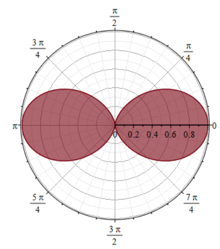
Properties of dipole radiation field for kr >>1:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right)\right)$$

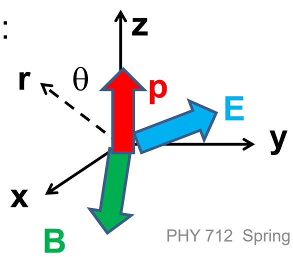
$$\tilde{\mathbf{B}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$



Example:



Note that vectors **r, E, B** are mutually orthogonal

Geometric properties of dipolar fields.

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Up to now, we have worked with the exact spherical harmonic expansion, evaluating the results in certain limits. Now consider analyzing the Green's function integral directly without use of Bessel functions. You may recognize this treatment as the Born approximation encountered in quantum mechanical scattering theory.

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

For $r>>r'$: $\quad |\mathbf{r}-\mathbf{r}'|\approx r-\hat{\mathbf{r}}\cdot\mathbf{r}'+...$

$$\tilde{\Phi}(\mathbf{r},\omega)\approx\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

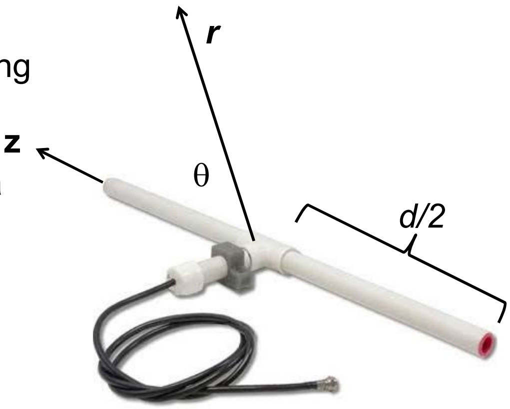
$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

➔Results equivalent to Bessel function expansion in the limit $kr \rightarrow \infty$.

This approach is similar to the Bessel function expansion if more terms were used.

Other radiation sources using ``alternative approach''

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx$$

$$\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\; e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\;\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}',\omega) = I_0 \sin\left(\frac{kd}{2} - k\,|\,z\,|\right)\delta(x)\delta(y)\hat{\mathbf{z}}$$

Up to now, we have been thinking about sources on the atomic scale. The analysis also works for macroscopic sources such as antennas. This example which follows your textbook is called a center fed antenna.

Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz'\, e^{-ik\cos(\theta)z'}\ \sin\left(\frac{kd}{2}-k|z'|\right)$$

$$=\hat{\mathbf{z}}\frac{\mu_0 I_0}{2\pi}\frac{e^{ikr}}{kr}\left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\ 2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

Analyzing the vector potential in this “Born” approximation, we obtain an analytic result for the radiation distribution.

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

for $kd=\pi$: $$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

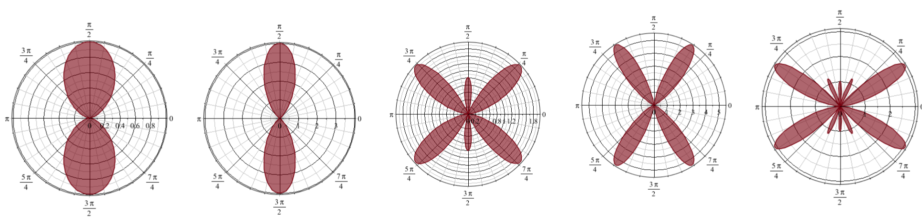
for $kd=2\pi$: $$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{4}{8\pi^2}\frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

The radiation pattern is quite sensitive to the relationship between the antenna length d and the wavelength of the radiation.

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$



kd= π 2π 3π 4π 5π

Here are some polar plots of the radiation patterns for various cases.

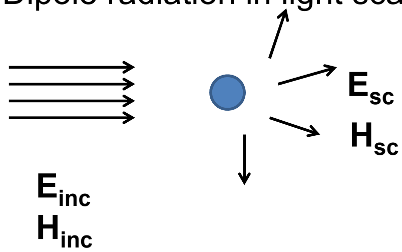
Another source of radiation –

Radiation due particles reacting to incident electromagnetic waves – scattering processes

Chapter 10 in **Jackson**

Now consider another radiation source – scattered light. We will introduce the topic (covered in Chapter 10) today, but discuss it more thoroughly on Friday.

Dipole radiation in light scattering by small (dielectric) particles



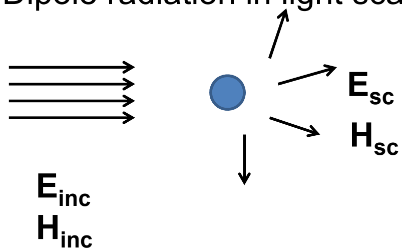
$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_\mathbf{0}\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_\mathbf{0} \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Imagine a plane wave of light incident on a sphere. The incident light produces oscillating dipoles within the sphere which in turn produce dipole radiation.

Dipole radiation in light scattering by small (dielectric) particles



Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{r^2\hat{\mathbf{r}}\cdot\left\langle \mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_{\mathbf{0}}\cdot\left\langle \mathbf{S}_{inc}\right\rangle_{avg}}$$

$$=\frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2}=\frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

The radiation depends on the initial polarization of the scattered polarization.

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section :

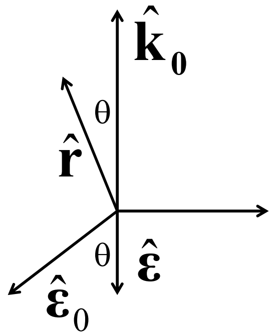
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

Using results from the electrostatic polarization analysis, we can deduce the polarization amplitude.

Scattering by dielectric sphere with permittivity ε and radius a:

For $\mathbf{E}_{\mathbf{inc}}$ polarized in scattering plane:

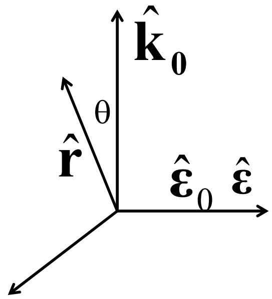


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right)== k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

We will continue this discussion on Friday, considering the geometrical effects.

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{\mathbf{inc}}$ polarized perpendicular to scattering plane:

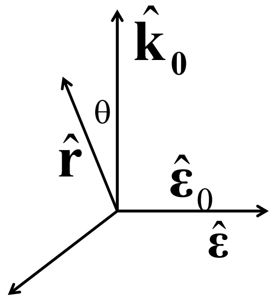
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

Assuming both polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$