PHY 712 Electrodynamics
12-12:50 AM Olin 103

Plan for Lecture 19:

Chap. 8 in Jackson – Wave Guides

1. **TEM, TE, and TM modes**
2. **Justification for boundary conditions; behavior of waves near conducting surfaces**

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Colloquium discussion: "How Can the Physics Colloquium Experience be Improved?"
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, February 26, 2020 at 3:00 PM

There will be a reception in the Olin Lounge at approximately 3:30 or 4 PM following the discussion. All interested persons are cordially invited to attend.

ABSTRACT
Over the years, the WFU Physics Colloquium series has hosted a variety of physicists, scientists and mathematicians in related fields to provide a window to new ideas. In principle, this opportunity for exposure to new ideas is an essential part of our education. In practice, there may be ways to improve the colloquium. Professor N. A. W. Holzwarth will lead a discussion among all students registered for PHY 301/601 and other interested participants on how we might optimize the colloquium experience.

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Next week --
Colloquium: "Changes in Blood Clot Structure and Mechanics in Cardiovascular and Thromboembolic Diseases"

Dr. Stephen Baker, Teacher Scholar Postdoctoral Fellow
WFU Physics

George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, March 4, 2020 at 3:00 PM

There will be a reception in the Olin Lounge at approximately 4 PM following the colloquium. All interested persons are cordially invited to attend.

ABSTRACT Studies in recent years have shown blood clot structure and mechanical properties to be a novel risk factor for cardiovascular diseases, the leading cause of morbidity and mortality worldwide. As a result, we need to better understand how the structural and mechanical properties of blood clots from patients with cardiovascular disease are different from those of healthy individuals. To study these properties, we need to determine how they change at different length scales. On the nano- and microscale, an atomic force microscope is an extremely versatile piece of equipment that can be used for nanometer to micrometer scale imaging, normal force unfolding of single molecules, or even novel lateral force techniques.

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13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces	#14	02/21/2020
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves	#15	02/24/2020
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves	#16	02/26/2020
18	Mon: 02/24/2020	Chap. 7	Optical effects of refractive indices		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		
	Fri: 03/13/2020	No class	*Spring Break*		
21	Mon: 03/16/2020	Chap. 9	Radiation from localized oscillating sources		

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Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E}; \quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

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Analysis of Maxwell's equations without sources -- continued:

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\nabla\times\left(\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\nabla^2\mathbf{B}-\mu\varepsilon\frac{\partial(\nabla\times\mathbf{E})}{\partial t}$$

$$=-\nabla^2\mathbf{B}+\mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

$$\nabla\times\left(\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\right)=-\nabla^2\mathbf{E}+\frac{\partial(\nabla\times\mathbf{B})}{\partial t}$$

$$=-\nabla^2\mathbf{E}+\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

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Analysis of Maxwell's equations without sources -- continued:

Both E and B fields are solutions to a wave equation:

$$\nabla^2\mathbf{B} - \frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2\mathbf{E} - \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2\frac{\mu_0\varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

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Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$

} For real ε, μ, n, k

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Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

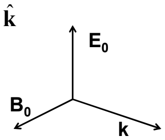
$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle\mathbf{S}\rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



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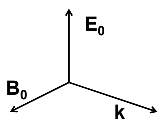
Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

TEM modes describe electromagnetic waves in lossless media and vacuum



For real ε, μ, n, k

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Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium : $\mathbf{D} = \varepsilon_b\mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{H} = 0$$

$$\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \nabla\times\mathbf{H} = \sigma\mathbf{E} + \varepsilon_b\frac{\partial\mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E},\mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i n_R(\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t}\right)$$

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Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0$$

$$-(n_R + i n_I)^2 + i\frac{\mu\sigma c^2}{\omega} + \mu\varepsilon_b c^2 = 0$$

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Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

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Some representative values of skin depth

Ref: Lorrain[2] and Corson

$$\frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

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Relative energies associated with field

Electric energy density: $\varepsilon_b|\mathbf{E}|^2$

Magnetic energy density: $\mu|\mathbf{H}|^2$

Ratio inside conducting media: $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu\left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b\mu\omega^2\delta^2}{2}$

$$=2\pi^2\frac{\varepsilon_b}{\varepsilon_0}\frac{\mu}{\mu_0}\frac{\delta^2}{\lambda^2}$$

For $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0|\mathbf{E}|^2}{\mu_0|\mathbf{H}|^2} = 1$

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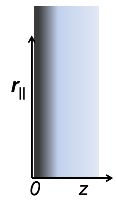
Fields near the surface on an ideal conductor -- continued

$$\text{For } \frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\text{In this limit, } \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + i n_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



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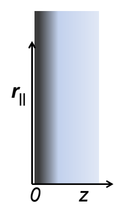
Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Note that the **H** field is larger than **E** field so we can write:

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

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Boundary values for ideal conductor

Inside the conductor:

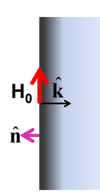
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$



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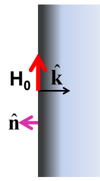
Wave guides – dielectric media with one or more metal boundary

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E}\big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H}\big|_S = 0$$



Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

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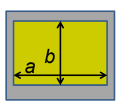
Analysis of rectangular waveguide

Boundary conditions at surface of waveguide:
$\mathbf{E}_{tangential}=0$, $\mathbf{B}_{normal}=0$

Cross section view



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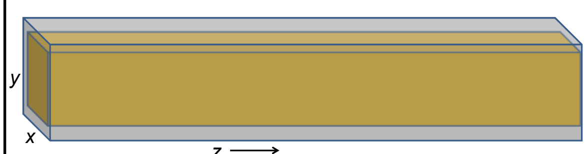
Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium: (assume ε to be real)

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} - \varepsilon\mu \frac{\partial \mathbf{E}}{\partial t} = 0$$

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Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)E_x(x,y) = 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$

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Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z=0.$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z=0.$$

$$\frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x. \qquad \frac{\partial B_z}{\partial y}-ikB_y=-i\mu\varepsilon\omega E_x.$$

$$ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y. \qquad ikB_x-\frac{\partial B_z}{\partial x}=-i\mu\varepsilon\omega E_y.$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z. \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-i\mu\varepsilon\omega E_z.$$

For TE mode with $E_z\equiv 0$

$$B_x=-\frac{k}{\omega}E_y$$

$$B_y=\frac{k}{\omega}E_x$$

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TE modes for rectangular wave guide continued:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x=\frac{\omega}{k}B_y=\frac{-i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y}=\frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b}B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y=-\frac{\omega}{k}B_x=\frac{i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x}=\frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a}B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$\mathbf{E}_{\text{tangential}}=0$ because: $E_z(x,y)\equiv 0$, $E_x(x,0)=E_x(x,b)=0$ and $E_y(0,y)=E_y(a,y)=0.$

$\mathbf{B}_{\text{normal}}=0$ because: $B_y(x,0)=B_y(x,b)=0$ and $B_x(0,y)=B_x(a,y)=0.$

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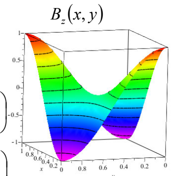
Solution for m=n=1

$$B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

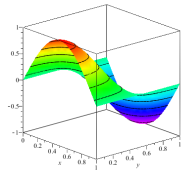
$$iE_x(x,y) = B_0\left(\frac{\omega n\pi / b}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x,y) = B_0\left(\frac{-\omega m\pi / a}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

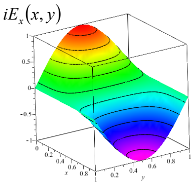
$B_z(x,y)$



$iE_y(x,y)$



$iE_x(x,y)$



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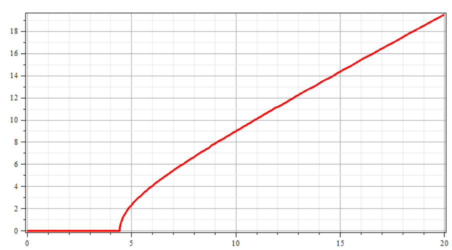
Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$



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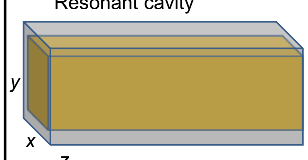
Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$B_i(x,y,z) = B_i(x,y)\sin(kz)$ or $B_i(x,y)\cos(kz)$

$$\Rightarrow k = \frac{p\pi}{d}$$

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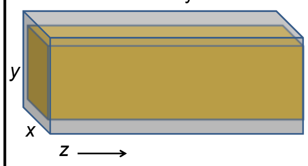
Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$
$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

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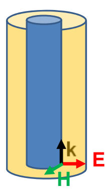
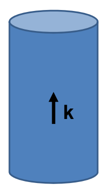
Wave guides – dielectric media with one or more metal boundary

Coaxial cable
TEM modes



Simple optical pipe
TE or TM modes



Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

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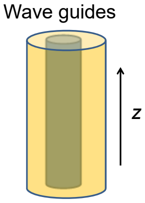
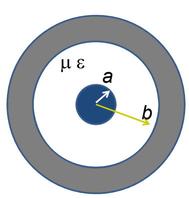
Wave guides



Coaxial cable
TEM modes

Top view:



Inside medium, μ ε assumed to be real

(following problem 8.2 in Jackson's text)

Maxwell's equations inside medium: for $a \le \rho \le b$

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \qquad \nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{B} = -i\omega\mu\varepsilon\mathbf{E} \qquad \nabla \cdot \mathbf{B} = 0$$

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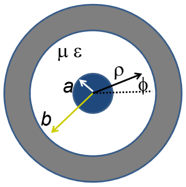
Electromagnetic waves in a coaxial cable -- continued

Top view:



Example solution for $a \le \rho \le b$

$$\mathbf{E} = \hat{\boldsymbol{\rho}} \Re\left(\frac{E_0 a}{\rho} e^{ikz - i\omega t} \right)$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \Re\left(\frac{B_0 a}{\rho} e^{ikz - i\omega t} \right)$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\, \hat{\mathbf{x}} + \sin\phi\, \hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}$$

Find:

$$k = \omega\sqrt{\mu\varepsilon}$$

$$E_0 = \frac{B_0}{\sqrt{\mu\varepsilon}}$$

Poynting vector within cable medium (with μ, ε):

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2\mu} \Re\left(\mathbf{E} \times \mathbf{B}^* \right) = \frac{|B_0|^2}{2\mu\sqrt{\mu\varepsilon}} \left(\frac{a}{\rho} \right)^2 \hat{\mathbf{z}}$$

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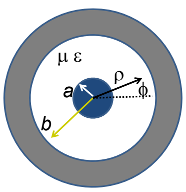
Electromagnetic waves in a coaxial cable -- continued

Top view:



Time averaged power in cable material:

$$\int_0^{2\pi} d\phi \int_a^b \rho d\rho \left(\langle \mathbf{S} \rangle_{avg} \cdot \hat{\mathbf{z}} \right) = \frac{|B_0|^2 \pi a^2}{\mu\sqrt{\mu\varepsilon}} \ln\left(\frac{b}{a} \right)$$

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