PHY 712 Electrodynamics
12-12:50 AM Olin 103

Plan for Lecture 18:

Complete reading of Chapter 7

1. **Comments on reflectivity of plane waves**
2. **Summary of complex response functions for electromagnetic fields**

1

13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces	#14	02/21/2020
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves	#15	02/24/2020
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves	#16	02/26/2020
18	Mon: 02/24/2020	Chap. 7	Optical effects of refractive indices		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		
	Fri: 03/13/2020	No class	*Spring Break*		
21	Mon: 03/16/2020	Chap. 9	Radiation from localized oscillating sources		

2

Some comments on the Fresnel Equations

1. Different behaviors of *s* and *p* polarization
2. Brewster's angle
3. Total internal reflection

3

Review: Electromagnetic plane waves in isotropic medium with real permeability and permittivity: μ ε.

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector for plane electromagnetic waves:

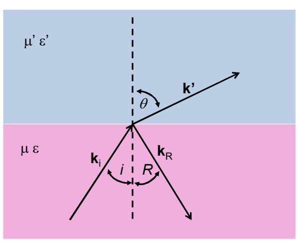
$$\langle\mathbf{S}\rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

4

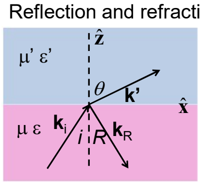
Review:
Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



$$n' = \varepsilon'\mu'$$
$$n = \varepsilon\mu$$
$$i = R$$
$$n\sin i = n'\sin\theta$$
$$|\mathbf{k}_i| = |\mathbf{k}_R| = n\frac{\omega}{c}$$
$$|\mathbf{k}'| = n'\frac{\omega}{c}$$

5

Review:
Reflection and refraction between two isotropic media



Reflectance, transmittance:

$$\mathrm{R} = \frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 \qquad \mathrm{T} = \frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}} = \left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

6

For s-polarization (E perpendicular to plane of incidence)

$$\frac{E_{0R}}{E_{0i}}=\frac{n\cos i-\frac{\mu}{\mu'}n'\cos\theta}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}\qquad \frac{E'_0}{E_{0i}}=\frac{2n\cos i}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

For p-polarization (E in plane of incidence)

$$\frac{E_{0R}}{E_{0i}}=\frac{\frac{\mu}{\mu'}n'^2\cos i-nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos i+nn'\cos\theta}\qquad \frac{E'_0}{E_{0i}}=\frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i+nn'\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

7

Reflectance for s-polarization

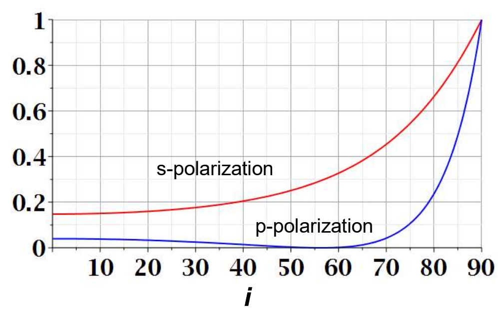
$$R_s=\left|\frac{E_{0R}}{E_{0i}}\right|^2=\left|\frac{n\cos i-\frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}{n\cos i+\frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}\right|^2$$

Reflectance for p-polarization

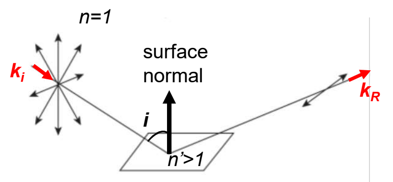
$$R_p=\left|\frac{E_{0R}}{E_{0i}}\right|^2=\left|\frac{\frac{\mu}{\mu'}n'\cos i-\frac{n}{n'}\sqrt{n'^2-n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i+\frac{n}{n'}\sqrt{n'^2-n^2\sin^2 i}}\right|^2$$

8

Example for $\mu=\mu'$; $n=1$ and $n'=1.5$



9



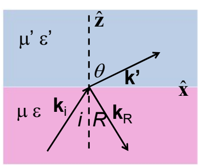
Polarization due to reflection from a refracting surface

Brewster's angle: for $i = i_B$, $R_p(i_B) = 0$

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2 \qquad \text{For } \mu' = \mu, \quad i_B = \tan^{-1}\left(\frac{n'}{n}\right)$$

10

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

Total internal reflection:

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

refracted field no longer propagates in medium $\mu'\varepsilon'$

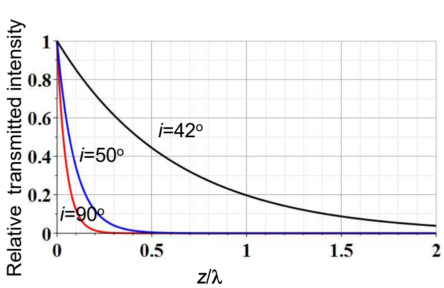
$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\,n\sqrt{\frac{\sin^2 i}{\sin^2 i_0} - 1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}_\parallel\cdot\mathbf{r} - ct)}\right)$$

11

Example of total internal reflection

$n'=1$ and $n=1.5$ ➔ $i_0 = \sin^{-1}(1/1.5) = 41.81°$



Transmitted illumination confined within a few wavelengths of the surface.

12

TIRF (total internal reflection fluorescence)

www.nikon.com/products/microscope-solutions/bioscience.../nikon_note_10_lr.pdf

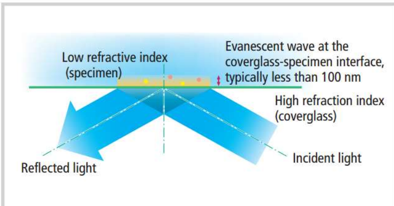


Figure 1: Creation of an evanescent wave at the coverglass-specimen interface

13

Design of TIRF device using laser and high power lens

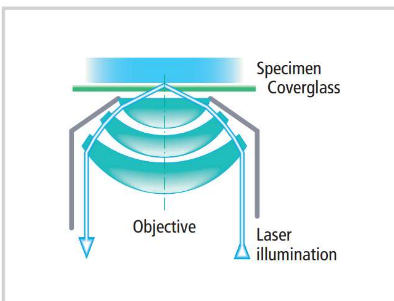
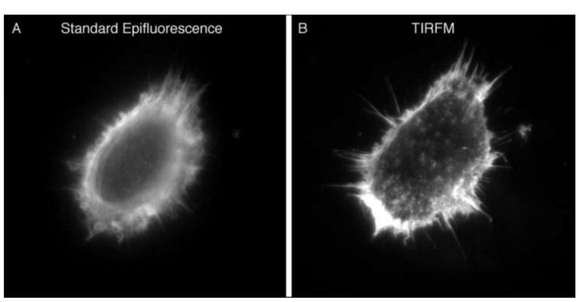


Figure 2: Through-the-lens laser TIRF.

14

PMC full text: Curr Protoc Cytom. Author manuscript; available in PMC 2015 Aug 18.
Published in final edited form as:
Curr Protoc Cytom. 2009 Oct; 0 12: Unit12.18.
doi: 10.1002/0471142956.cy1218s50

Figure 1



15

Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n' + n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 16

16

Extension to complex refractive index $n = n_R + i\, n_I$

Suppose $\mu = \mu'$, $n = \text{real}$, $n' = n'_R + in'_I$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2 = \frac{(n'_R - n)^2 + (n'_I)^2}{(n'_R + n)^2 + (n'_I)^2}$$

Note that for $n'_I >> |n'_R \pm n|$:

$$\mathrm{R} \approx 1$$

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 17

17

Origin of imaginary contributions to permittivity --
Review: Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

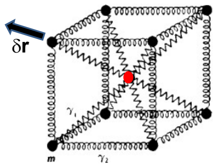
$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2}$$

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 18

18

Extensions of the Drude model for lattice vibrations



In principle, the ideas of the Drude model apply both to the ionic vibrations which occur at low frequency (~10^{12} Hz) contributing to the so called static permittivity function ε_s and to the electronic vibrations which occur at high frequency (~10^{15} Hz) contributing to the so called high frequency permittivity function ε_∞.

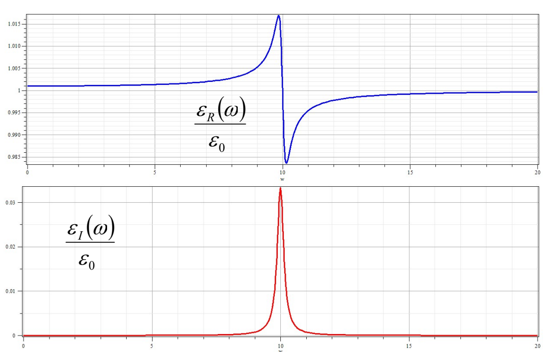
In this model at high frequencies, only the electrons contribute to the polarization: $\epsilon_\infty = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|}$

At low frequencies both electrons and ions contribute to the polarization: $\epsilon_s = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|} + \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|}$

$\Rightarrow \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|} = \epsilon_s - \epsilon_\infty$

19

Drude model dielectric function:



20

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{For } \omega >> \omega_i \quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

21

Analysis for Drude model dielectric function – continued --

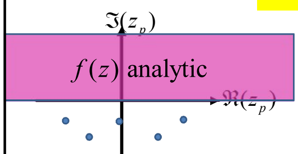
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \quad \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$



22

Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1\right) \frac{1}{\omega' - \omega}$$

with $\quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$

23

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

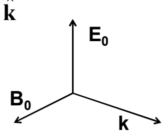
$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle \mathbf{S} \rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

$$\langle u \rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



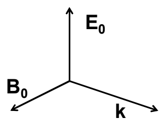
24

Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)\qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

TEM modes describe electromagnetic waves in lossless media and vacuum



For real
ε, μ, n, k

25

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D}=\varepsilon_b\mathbf{E}$ $\quad \mathbf{J}=\sigma\mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\cdot\mathbf{H}=0$$

$$\nabla\times\mathbf{E}=-\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \nabla\times\mathbf{H}=\sigma\mathbf{E}+\varepsilon_b\frac{\partial\mathbf{E}}{\partial t}$$

$$\left(\nabla^2-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{F}=0 \qquad \mathbf{F}=\mathbf{E},\mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k}=(n_R+in_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t)=e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{in_R(\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t}\right)$$

26

Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k}=(n_R+in_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=0$$

$$-(n_R+in_I)^2+i\frac{\mu\sigma c^2}{\omega}+\mu\varepsilon_b c^2=0$$

27

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 28

28

Some representative values of skin depth

Ref: Lorrain[2] and Corson

$$\frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 29

29

Relative energies associated with field

Electric energy density: $\varepsilon_b|\mathbf{E}|^2$

Magnetic energy density: $\mu|\mathbf{H}|^2$

Ratio inside conducting media: $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu\left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b\mu\omega^2\delta^2}{2}$

$$= 2\pi^2\frac{\varepsilon_b}{\varepsilon_0}\frac{\mu}{\mu_0}\frac{\delta^2}{\lambda^2}$$

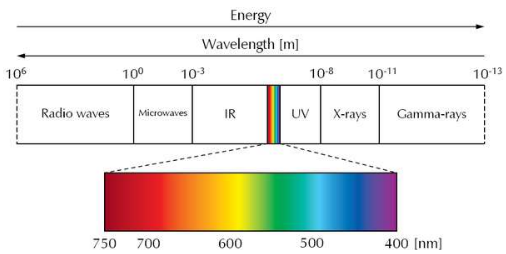
For $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0|\mathbf{E}|^2}{\mu_0|\mathbf{H}|^2} = 1$

02/24/2020 PHY 712 Spring 2020 -- Lecture 18 30

30

Various wavelengths λ --



02/24/2020 PHY 712 Spring 2020 -- Lecture 18 31

31