PHY 712 Electrodynamics
12-12:50 AM Olin 103

Plan for Lecture 17:

Continue reading Chapter 7

1. **Real and imaginary contributions to electromagnetic response**
2. **Frequency dependence of dielectric materials; Drude model**
3. **Kramers-Kronig relationships**

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11	Fri: 02/07/2020	Chap. 5	Magnetostatics	#10	02/12/2020
12	Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	02/14/2020
13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces	#14	02/21/2020
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves	#15	02/24/2020
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves	#16	02/26/2020
18	Mon: 02/24/2020	Chap. 7	Refractive index		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		
	Fri: 03/13/2020	No class	*Spring Break*		
21	Mon: 03/16/2020	Chap. 9	Radiation from localized oscillating sources		

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Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E}; \quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

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Plane wave solutions to sourceless Maxwell's equations; extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex:

$$\varepsilon = \varepsilon_R + i\varepsilon_I \qquad \frac{\varepsilon}{\varepsilon_0} = \left(n_R + i n_I\right)^2 \equiv \alpha + i\beta$$

$$n_R^2 - n_I^2 = \alpha \qquad 2 n_R n_i = \beta$$

$$n_I = \sqrt{n_R^2 - \alpha} \qquad 4n_R^4 - 4n_R^2\alpha - \beta^2 = 0$$

$$n_R = \left(\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}\right)^{1/2} \qquad n_I = \left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}\right)^{1/2}$$

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n_R\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) e^{-\frac{\omega}{c}n_I\hat{\mathbf{k}}\cdot\mathbf{r}}$$

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Paul Karl Ludwig Drude 1863-1906

http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

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Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

$$\text{For} \quad \delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}, \qquad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$$

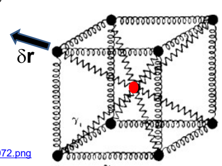
Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}e^{-i\omega t}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$



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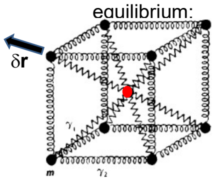
Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

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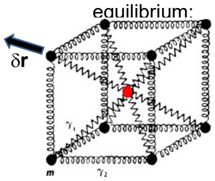
Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

For $\delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}$, $\quad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}$$

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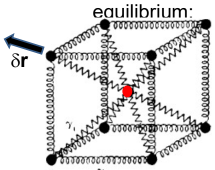
Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N\sum_i f_i\ \mathbf{p}_i$$

$N \equiv$ number of dipoles/volume

$f_i \equiv$ fraction of type i dipoles

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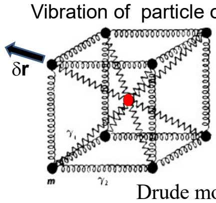
Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \varepsilon\mathbf{E} \ = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + N\sum_i f_i \ \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \ \delta\mathbf{r} = \frac{q_i^2\mathbf{E}_0}{m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}e^{-i\omega t}$$

$$\varepsilon\mathbf{E} = \ \varepsilon_0\mathbf{E}_0 e^{-i\omega t}\left(1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}\right)$$

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Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^2}$$

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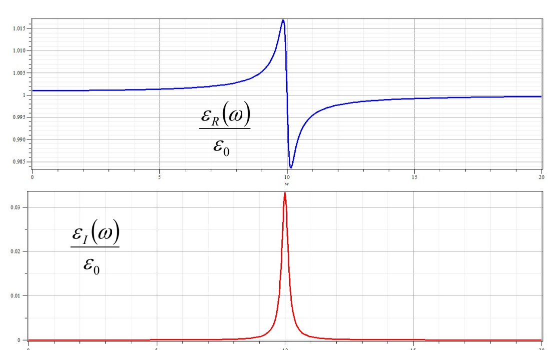
Drude model dielectric function:



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Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

For $\omega >> \omega_i$ $\quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

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Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_{i>0} f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i} + iNf_0\frac{q_0^2}{\varepsilon_0 m_0}\frac{1}{\omega(\gamma_0-i\omega)}$$

$$\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i\frac{\sigma(\omega)}{\varepsilon_0\omega}$$

Some details:

$\mathbf{D} = \varepsilon_b\mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0\frac{q_0^2}{m_0}\frac{1}{(\gamma_0 - i\omega)}$$

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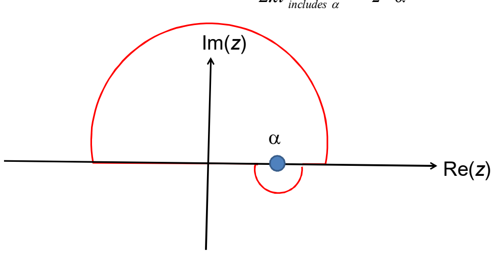
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad f(\alpha) = \frac{1}{2\pi i}\oint_{includes\ \alpha} dz\, \frac{f(z)}{z-\alpha}$$



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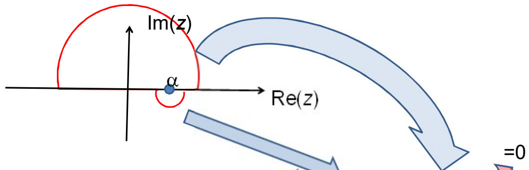
Kramers-Kronig transform -- continued



$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\ \alpha} dz\,\frac{f(z)}{z-\alpha}=\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\int_{rest} dz\,\frac{f(z)}{z-\alpha}\right)$$

$$f(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

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Kramers-Kronig transform -- continued

$$f(\alpha)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Suppose $f(z_R)=f_R(z_R)+if_I(z_R)$:

$$\Rightarrow\ \frac{1}{2}\left(f_R(\alpha)+if_I(\alpha)\right)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)+if_I(z_R)}{z_R-\alpha}$$

$$\Rightarrow\ f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R-\alpha}$$

$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R-\alpha}$$

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Kramers-Kronig transform -- continued

$$f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R-\alpha}$$

$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R-\alpha}$$

This Kramers-Kronig transform is useful for the dielectric function

when $f(z_R)\Rightarrow\frac{\varepsilon(\omega)}{\varepsilon_0}-1$

Must show that: 1. $f(z)$ is analytic for $z_I>0$

2. $f(z)$ vanishes for $z\to\infty$

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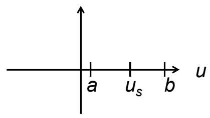

Some practical considerations

Principal parts integration :

$$P\int_a^b du\, g(u) = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, g(u) + \int_{u_s+\upsilon}^{b} du\, g(u) \right)$$

Example :

$$P\int_a^b du\, \frac{1}{u\text{-}u_s} = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, \frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\, \frac{1}{u\text{-}u_s} \right)$$

$$= \lim_{\upsilon \to 0}\left(\ln\left(\frac{\upsilon}{u_s - a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right) \right) = \ln\left(\frac{b-u_s}{u_s - a}\right)$$

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More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$
$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$
$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I > 0$)

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Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right) \Rightarrow \text{vanishes at large } z$$

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Analysis for Drude model dielectric function – continued --
Analytic properties:

$$f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-{z_P}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P)\leq 0 \quad\Rightarrow f(z)$ is analytic for $\Im(z_P)>0$

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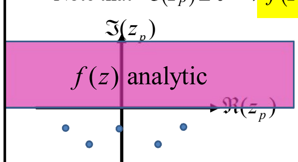
$$f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-{z_P}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P)\leq 0 \quad\Rightarrow f(z)$ is analytic for $\Im(z_P)>0$



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Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0}-1=\frac{1}{\pi}P\int_{-\infty}^{\infty}d\omega'\frac{\varepsilon_I(\omega')}{\varepsilon_0}\frac{1}{\omega'-\omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0}=-\frac{1}{\pi}P\int_{-\infty}^{\infty}d\omega'\left(\frac{\varepsilon_R(\omega')}{\varepsilon_0}-1\right)\frac{1}{\omega'-\omega}$$

with $\varepsilon_R(-\omega)=\varepsilon_R(\omega);\quad \varepsilon_I(-\omega)=-\varepsilon_I(\omega)$

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Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\left\{\mathbf{E}(\mathbf{r},t)+\int_0^\infty d\tau\ G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\int_0^\infty d\tau\ G(\tau)e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$f(t)=\int_{-\infty}^{\infty} g(t')h(t-t')dt'$ where the functions $f(t), g(t)$, and $h(t)$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega)=\int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt' \qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}d\omega$$

It follows that: $\tilde{f}(\omega)=\tilde{g}(\omega)\tilde{h}(\omega)$

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Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\left\{\mathbf{E}(\mathbf{r},t)+\int_0^\infty d\tau\ G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega \qquad \tilde{G}(\omega)=\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\int_0^\infty d\tau\ G(\tau)e^{i\omega\tau}$$

For $\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$

$$G(\tau)=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i\equiv\sqrt{\omega_i^2-\gamma_i^2/4}$

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Some details

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

Let $f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2-z_P{}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2} \quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

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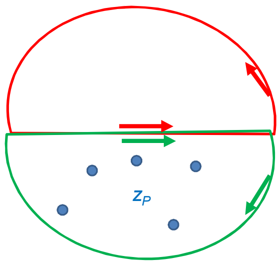
$$G(\tau) = \frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz = i\sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau}e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}$$

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$$G(\tau) = \frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz = i\sum_P \text{Res}(z_P)$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2 - z_P{}^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2/4}$ assuming $\omega_i^2 - \gamma_i^2/4 \geq 0$

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