

**PHY 712 Electrodynamics**  
**12-12:50 AM MWF Olin 103**

**Plan for Lecture 14:**

**Start reading Chapter 6**

- 1. Maxwell's full equations; effects of time varying fields and sources**
- 2. Gauge choices and transformations**
- 3. Green's function for vector and scalar potentials**

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**Course schedule for Spring 2020**  
(Preliminary schedule -- subject to frequent adjustment.)

Lecture date	JDJ Reading	Topic	HW	Due date
1 Mon: 01/13/2020	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/17/2020
2 Wed: 01/15/2020	Chap. 1	Electrostatic energy calculations	#2	01/22/2020
3 Fri: 01/17/2020	Chap. 1	Electrostatic potentials and fields	#3	01/24/2020
Mon: 01/20/2020	No class	Martin Luther King Holiday		
4 Wed: 01/22/2020	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	01/27/2020
5 Fri: 01/24/2020	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/31/2020
6 Mon: 01/27/2020	Chap. 2 & 3	Image charge constructions	#6	02/03/2020
7 Wed: 01/29/2020	Chap. 2 & 3	Cylindrical and spherical geometries	#7	02/05/2020
8 Fri: 01/31/2020	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/07/2020
9 Mon: 02/03/2020	Chap. 4	Dipoles and Dielectrics	#9	02/10/2020
10 Wed: 02/05/2020	Chap. 4	Polarization and Dielectrics		
11 Fri: 02/07/2020	Chap. 5	Magnetostatics	#10	02/12/2020
12 Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	02/14/2020
13 Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14 Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15 Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces		
16 Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves		
17 Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves		
18 Mon: 02/24/2020	Chap. 7	Refractive index		
19 Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20 Fri: 02/28/2020	Chap. 1-8	Review		

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Full electrodynamics with time varying fields and sources

## Maxwell's equations



*"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"*

Image of statue of James Clerk-Maxwell in Edinburgh      Richard P Feynman

<http://www.clerkmaxwellfoundation.org/>

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## Maxwell's equations

Coulomb's law :  $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law :  $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law :  $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles :  $\nabla \cdot \mathbf{B} = 0$

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## Maxwell's equations

Microscopic or vacuum form ( $\mathbf{P} = 0$ ;  $\mathbf{M} = 0$ ):

Coulomb's law :  $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law :  $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law :  $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles :  $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

or  $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left( \frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left( \frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Coulomb gauge form -- require  $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that  $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_t$  with  $\nabla \times \mathbf{J}_i = 0$  and  $\nabla \cdot \mathbf{J}_t = 0$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form -- require  $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that  $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_t$  with  $\nabla \times \mathbf{J}_i = 0$  and  $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_i = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_t = -\epsilon_0 \nabla \cdot \frac{\partial(\nabla \Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \epsilon_0 \mu_0 \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}_t$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Review of the general equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left( \frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Lorentz gauge form -- require  $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require  $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials :  $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$  and  $\Phi'_L = \Phi_L - \frac{\partial\Lambda}{\partial t}$

Yields same physics provided that :  $\nabla^2\Lambda - \frac{1}{c^2} \frac{\partial^2\Lambda}{\partial t^2} = 0$

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Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2\Phi_L - \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = -\rho / \epsilon_0$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation :

$$\nabla^2\Psi - \frac{1}{c^2} \frac{\partial^2\Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r}, t) \Rightarrow$  wave field  $f(\mathbf{r}, t) \Rightarrow$  source

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Let  $\Psi$  represent  $\Phi, A_x, A_y, A_z$  Let  $f$  represent  $\rho, J_x, J_y, J_z$

$$\nabla^2 \Psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Psi(\mathbf{r}, t)}{\partial t^2} = -4\pi f(\mathbf{r}, t)$$

Green's function :

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Formal solution for field  $\Psi(\mathbf{r}, t)$ :

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' G(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function :

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

For the case of isotropic boundary values at infinity :

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left( t' - \left( t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right) \right)$$

Formal solution for field  $\Psi(\mathbf{r}, t)$ :

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left( t' - \left( t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right) \right) f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function:

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

"Proof" -- Fourier analysis in the time domain -- note that

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \tilde{G}(\mathbf{r}, \mathbf{r}', \omega)$$

$$\Rightarrow \left( \nabla^2 + \frac{\omega^2}{c^2} \right) \tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}(\mathbf{r} - \mathbf{r}', \omega)$$

Further assuming that  $\tilde{G}(\mathbf{r} - \mathbf{r}', \omega)$  is isotropic in  $|\mathbf{r} - \mathbf{r}'| \equiv R$ :

$$\left(\frac{1}{R} \frac{d^2}{dR^2} R + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

$$\text{Solution: } \tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{R} e^{\pm i\omega R/c}$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} e^{\pm i\omega|\mathbf{r} - \mathbf{r}'|/c}$$

$$\begin{aligned} G(\mathbf{r}, t; \mathbf{r}', t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \tilde{G}(\mathbf{r}, \mathbf{r}', \omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \frac{1}{|\mathbf{r} - \mathbf{r}'|} e^{\pm i\omega|\mathbf{r} - \mathbf{r}'|/c} \\ &= \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t' \pm |\mathbf{r} - \mathbf{r}'|/c)} \right) \\ &= \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - t \mp |\mathbf{r} - \mathbf{r}'|/c) \end{aligned}$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t \pm |\mathbf{r} - \mathbf{r}'|/c\right)\right)$$

Solution for field  $\Psi(\mathbf{r}, t)$ :

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --  
 Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz *The Classical Theory of Fields*, Chapter 8.)

Consider the fields produced by the following source: a point charge  $q$  moving on a trajectory  $\mathbf{R}_q(t)$ .

Charge density:  $\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density:  $\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$ , where  $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$ .



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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iint d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \iint d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)).$$

We performing the integrations over first  $d^3r'$  and then  $dt'$  making use of the fact that for any function of  $t'$ ,

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t_r)|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

Notation:  $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$   $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

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**Comment on Lienard-Wiechert potential results**

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

Note that for any function  $F(x)$ :

$$\int_{-\infty}^{\infty} F(x) \delta(x - x_0) dx = F(x_0)$$

Now consider a function  $p(x)$ , for which  $p(x_i) = 0$  for  $i = 1, 2, \dots$

$$\int_{-\infty}^{\infty} F(x) \delta(p(x)) dx = \int_{-\infty}^{\infty} F(x) \left( \sum_i \delta\left(x - x_i \frac{dp}{dx}\bigg|_{x_i}\right) \right) dx$$

$$= \sum_i \frac{F(x_i)}{\left| \frac{dp}{dx} \right|_{x_i}}$$

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**Comment on Lienard-Wiechert potential results -- continued**

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

In this case we have:  $\int_{-\infty}^{\infty} f(t') \delta(p(t')) dt' = \frac{f(t_r)}{\left| 1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|} \right|}$

where:  $p(t') \equiv t' - \left( t - \frac{|\mathbf{r} - \mathbf{R}_q(t')|}{c} \right)$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'} \cdot (\mathbf{r} - \mathbf{R}_q(t'))}{c|\mathbf{r} - \mathbf{R}_q(t')|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t') \cdot (\mathbf{r} - \mathbf{R}_q(t'))}{c|\mathbf{r} - \mathbf{R}_q(t')|}$$

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**Summary of results for fields due to moving charge – Liénard Wiechert potentials**

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

Notation:  $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$   $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$$

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