PHY 712 Electrodynamics
12-12:50 AM MWF Olin 103

Plan for Lecture 13:

Finish reading Chapter 5

1. **Recap of hyperfine interaction**
2. **Macroscopic magnetization density M**
3. **H field and its relation to B**
4. **Magnetic boundary values**

1

Colloquium: "Ghosts from Dying Stars: Using Neutrinos to Probe Supernovae and Beyond"

Dr. Samuel Flynn
WFU Alum and NCSU Graduate Student
Theoretical Nuclear & Particle Physics
NC State University
Raleigh, NC
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, February 12, 2020 at 3:00 PM

There will be a reception in the Olin Lounge at approximately 4 PM following the colloquium. All interested persons are cordially invited to attend.

2

Course schedule for Spring 2020

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2020	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/17/2020
2	Wed: 01/15/2020	Chap. 1	Electrostatic energy calculations	#2	01/22/2020
3	Fri: 01/17/2020	Chap. 1	Electrostatic potentials and fields	#3	01/24/2020
	Mon: 01/20/2020	No class	Martin Luther King Holiday		
4	Wed: 01/22/2020	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	01/27/2020
5	Fri: 01/24/2020	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/31/2020
6	Mon: 01/27/2020	Chap. 2 & 3	Image charge constructions	#6	02/03/2020
7	Wed: 01/29/2020	Chap. 2 & 3	Cylindrical and spherical geometries	#7	02/05/2020
8	Fri: 01/31/2020	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/07/2020
9	Mon: 02/03/2020	Chap. 4	Dipoles and Dielectrics	#9	02/10/2020
10	Wed: 02/05/2020	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/07/2020	Chap. 5	Magnetostatics	#10	02/12/2020
12	Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	02/14/2020
13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations		
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces		
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves		
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/24/2020	Chap. 7	Refractive index		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		

3

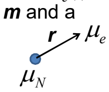
Summary of hyperfine interaction form:

Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole ***m*** and a magnetic field **B:** $E_{int} = -\mathbf{m}\cdot\mathbf{B}$



In this case: $E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$

$$\mathbf{B}_{\boldsymbol{\mu}_e}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_e\cdot\hat{\mathbf{r}}) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_e\,\delta^3(\mathbf{r})\right\}$$

4

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Here we assume that nuclear position is **r**=*0.*

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ in the vicinity of the nucleus $(\mathbf{r}\to 0)$.

5

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla\times\mathbf{A}_{\mathbf{J}_e}\Big|_{\mathbf{r}\to 0} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \nabla\times\frac{\hat{\mathbf{z}}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}')\times(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}'\times(\hat{\mathbf{z}}\times\mathbf{r}')}{r'^3}\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'}$$

$$\hat{\mathbf{r}}'\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}}') = \hat{\mathbf{z}}(\mathbf{1}-\cos^2\boldsymbol{\theta}') - \hat{\mathbf{x}}\cos\theta'\sin\theta'\cos\phi' - \hat{\mathbf{y}}\cos\theta'\sin\theta'\sin\phi'.$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}}r'^2\sin^2\theta'}{r'^3}\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\,\hat{\mathbf{z}}\int d^3r' \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\,\hat{\mathbf{z}}\left\langle\frac{1}{r'^3}\right\rangle$$

6

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle\frac{3(\boldsymbol{\mu}_{\mathrm{N}}\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_{\mathrm{e}}\cdot\hat{\mathbf{r}})-\boldsymbol{\mu}_{\mathrm{N}}\cdot\boldsymbol{\mu}_{\mathrm{e}}}{r^3}+\frac{8\pi}{3}\boldsymbol{\mu}_{\mathrm{N}}\cdot\boldsymbol{\mu}_{\mathrm{e}}\delta^3(\mathbf{r})\right\rangle+\frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\boldsymbol{\mu}_{\mathrm{N}}}{r^3}\right\rangle\right).$$

In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

7

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int d^3r\, \mathbf{r}\times\mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of its **J**(**r**).

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3}$$

8

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i\delta^3(\mathbf{r}-\mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ and macroscopic magnetization **M**(**r**). Note: the designation $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}+\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

9

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \right)$$

Note that:

$$\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

10

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that for the case that $\nabla\cdot\mathbf{A} = 0$:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \nabla\times(\nabla\times\mathbf{A}(\mathbf{r})) = -\nabla^2\mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')\right)$$

$$= \mu_0\left(\mathbf{J}_{free}(\mathbf{r}) + \nabla\times\mathbf{M}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

11

Magnetic field contributions

$$\nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

Define "new" magnetic field vector:

$$\mu_0\mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0\mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

12

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$U = -\mathbf{m}\cdot\mathbf{B}$$

Macroscopic energies --

It can be shown that: $W_B = \frac{1}{2}\int d^3r\ \mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})$

In analogy to: $W_E = \frac{1}{2}\int d^3r\ \mathbf{E}(\mathbf{r})\cdot\mathbf{D}(\mathbf{r})$

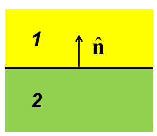
13

Summary of equations of magnetostatics:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}_{total}(\mathbf{r})$$
$$\nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r})+\mathbf{M}(\mathbf{r}))$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla\times\mathbf{H}(\mathbf{r}) = 0$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$



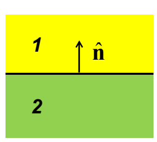
14

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla\times\mathbf{H}(\mathbf{r}) = 0$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

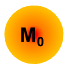
At boundary:

$$\mathbf{H}_1\times\hat{\mathbf{n}} = \mathbf{H}_2\times\hat{\mathbf{n}}$$
$$\mathbf{B}_1\cdot\hat{\mathbf{n}} = \mathbf{B}_2\cdot\hat{\mathbf{n}}$$



15

Example magnetostatic boundary value problem

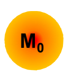


$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$
$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

16

Example magnetostatic boundary value problem -- continued

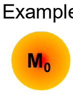


$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$
$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi}\int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$
$$= -\frac{1}{4\pi}\int d^3r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right]$$
$$= -\frac{1}{4\pi}\nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

17

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases} \qquad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi}\nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

For this example:

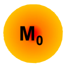
$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\left(4\pi\int_0^a r'^2\, dr' \frac{1}{r_>}\right)$$

For $r \le a$: $\quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z}\left(\frac{a^2}{2} - \frac{r^2}{6}\right) = \frac{M_0 z}{3}$

For $r > a$: $\quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z}\left(\frac{a^3}{3r}\right) = \frac{M_0 a^3 z}{3r^3}$

18

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

For $r \le a$: $\Phi_H(\mathbf{r}) = \frac{M_0 z}{3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0}{3}\hat{\mathbf{z}}$

For $r > a$: $\Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 19

19

Check boundary values:

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{H}(a\hat{\mathbf{r}})\times\hat{\mathbf{r}} = -\frac{M_0}{3}\hat{\mathbf{z}}\times\hat{\mathbf{r}}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{H}(a\hat{\mathbf{r}})\times\hat{\mathbf{r}} = -\frac{M_0 a^3}{3}\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{a^3}$$

For $r \le a$: $\mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(a\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = \mu_0\frac{2M_0}{3}\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}$

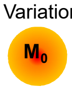
For $r > a$: $\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

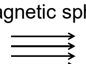
$$\mathbf{B}(a\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = -\mu_0\frac{M_0 a^3}{3}\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}\left(\frac{1}{a^3} - \frac{3a^2}{a^5}\right)$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 20

20

Variation; magnetic sphere plus external field $\mathbf{B}_0$





$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \le a \\ 0 & r > a \end{cases}$$

By superposition:

For $r \le a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0\frac{2}{3}\mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0\mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 21

21

Summary of equations of magnetostatics:

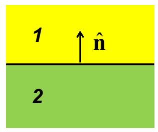
$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$
$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$
$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



22

Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism:

$$\mathbf{B} = \mu \mathbf{H}$$

$\mu > \mu_0 \Rightarrow$ paramagnetic material

$\mu < \mu_0 \Rightarrow$ diamagnetic material

For ferromagnetic, antiferromagnetic materials

$\mathbf{B} = f(\mathbf{H})$ (with hysteresis)

23

https://en.wikipedia.org/wiki/Permeability_(electromagnetism)

Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, max., μ/μ_0	Magnetic field
Metglas 2714A (annealed)		1.26×10^{0}	1 000 000[10]	At 0.5 T
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000[11]	
NANOPERM®		1.0×10^{-1}	80 000[12]	At 0.5 T
Mu-metal		2.5×10^{-2}	20 000[13]	At 0.002 T
Mu-metal		6.3×10^{-2}	50 000[14]	
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000[15]	
Permalloy	8000	1.0×10^{-2}	8000[13]	At 0.002 T

24

Mumetal Magnetic Shielding

Mu metal is a soft ferromagnetic alloy that has extremely high initial and maximum magnetic permeability. It is used in electric transformer, storage disks, magnetic phonographs, resonance devices and superconducting circuits.

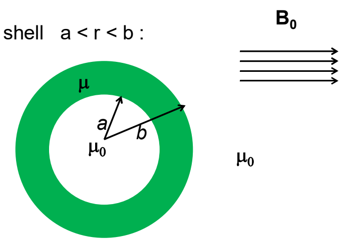
Mumetal alloy generally attributes relative permeability about 80,000 to 100,000 than the normal steel alloy. It is also called as soft magnetic alloy and offers low magnetic anisotropy and magnetostriction providing low corceivity to saturate the low magnetic fields. It provides nominal hysteresis loses when the alloy is employed in the AC magnetic circuits.

Composed of 80% Ni, 15% Fe, 5% Mo+other materials

25

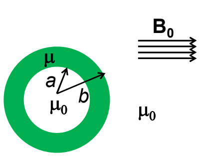
Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



26

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



For this case:

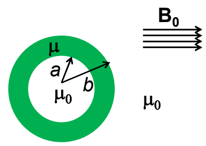
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$
$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries:

$$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$$
$$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$$

27

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



$$\text{Let:}\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r})$$

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0 \qquad \Rightarrow \nabla^2\Phi_H(\mathbf{r}) = 0$$

$$\text{For}\quad 0 \le r \le a \qquad \Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$$

$$\text{For}\quad a \le r \le b \qquad \Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right) P_l(\cos\theta)$$

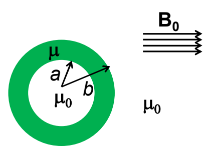
$$\text{For}\quad r \ge b \qquad \Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 28

28

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions (only $l=1$ terms contribute):



$$\text{At}\quad r=a \qquad \delta_1 = \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right)$$

$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

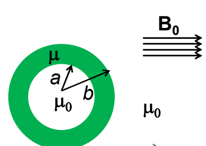
$$\text{At}\quad r=b \qquad \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{b^3}\right) = -\frac{B_0}{\mu_0} - 2\frac{\alpha_1}{b^3}$$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b\frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 29

29

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0+1)(\mu/\mu_0+2) - 2(a/b)^3(\mu/\mu_0-1)^2}\right)\frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0}\left(\frac{-9/2}{\left(1-(a/b)^3\right)}\frac{B_0}{\mu_0}\right)$$

02/12/2020 PHY 712 Spring 2020 -- Lecture 13 30

30