PHY 712 Electrodynamics
12-12:50 AM MWF Olin 103

Plan for Lecture 11:

Start reading Chapter 5

A. Magnetostatics

B. Vector potential

C. Example: current loop

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Course schedule for Spring 2020

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2020	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/17/2020
2	Wed: 01/15/2020	Chap. 1	Electrostatic energy calculations	#2	01/22/2020
3	Fri: 01/17/2020	Chap. 1	Electrostatic potentials and fields	#3	01/24/2020
	Mon: 01/20/2020	No class	Martin Luther King Holiday		
4	Wed: 01/22/2020	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	01/27/2020
5	Fri: 01/24/2020	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/31/2020
6	Mon: 01/27/2020	Chap. 2 & 3	Image charge constructions	#6	02/03/2020
7	Wed: 01/29/2020	Chap. 2 & 3	Cylindrical and spherical geometries	#7	02/05/2020
8	Fri: 01/31/2020	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/07/2020
9	Mon: 02/03/2020	Chap. 4	Dipoles and Dielectrics	#9	02/10/2020
10	Wed: 02/05/2020	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/07/2020	Chap. 5	Magnetostatics	#10	02/12/2020
12	Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields		
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations		
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces		
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves		
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/24/2020	Chap. 7	Refractive index		

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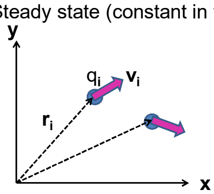
Magnetostatics

Magnetic flux density or magnetic induction field **B**
Steady state (constant in time) current density **J**



$$\mathbf{J}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Note that "statics" implies that $\nabla \cdot \mathbf{J} = 0$.

This follows from the continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

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Comparison of electrostatics and magnetostatics

Electrostatic field due to charge density $\rho(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

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Alternative forms magnetostatic equations

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0}{4\pi}\int d^3r'\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{J}(\mathbf{r}')$$

Note that: $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$

Also note that: $\nabla\times\left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) = \nabla s(\mathbf{r})\times\mathbf{V}(\mathbf{r}) + s(\mathbf{r})\nabla\times\mathbf{V}(\mathbf{r})$

let $s(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ and $\mathbf{V}(\mathbf{r}) = \mathbf{J}(\mathbf{r}')$, where $\mathbf{r}'$ is fixed

$$\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{J}(\mathbf{r}') = \nabla s(\mathbf{r})\times\mathbf{V}(\mathbf{r}) = \nabla\times\left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) - s(\mathbf{r})\nabla\times\mathbf{V}(\mathbf{r})$$

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Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{J}(\mathbf{r}')$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\left(\nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\times\mathbf{J}(\mathbf{r}')\right)$$

$$\Rightarrow \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

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Alternative forms magnetostatic equations -- continued

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$

"Proof" of Ampere's law for magnetostatic system:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that: $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$

Recall that: $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$ and $\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$

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Differential forms of magnetostatic equations:

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$

Magnetostatic vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \nabla s(\mathbf{r})$$

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Non uniqueness of the magnetostatic vector potential

Note that: $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r})$

if $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla s(\mathbf{r})$

Example: for $\mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2} B_0 (x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$$

or $\mathbf{A}(\mathbf{r}) = B_0 x \hat{\mathbf{y}}$

or $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

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Differential form of Ampere's law in terms of vector potential:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\text{If} \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \quad \text{(Coulomb gauge)} \quad \Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

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Comment on determining $\mathbf{A}(\mathbf{r})$ in the Coulomb gauge:

Differential form: $\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$

Integral form: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

PHY 712 -- Assignment #10

February 7, 202

Start reading Chapter 5 in **Jackson** .

1. Consider an infinitely long cylindrical wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically, in terms of *r* the radial parameter of the cylindrical coordinates of the system the current density is $\mathbf{J}(r)=\mathbf{J}_0$, where $\mathbf{J}_0$ is a constant vector along the z-axis, for $r \leq a$ and zero otherwise.
   a. Find the vector potential (**A**) for all *r*.
   b. Find the magnetic flux field (**B**) for all *r*.

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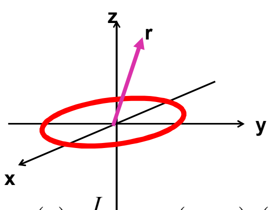
Magnetostatics example: current loop



$$\mathbf{J}(\mathbf{r}') = \frac{I}{a} \sin\theta' \delta(\cos\theta') \delta(r'-a)(-\sin\phi' \hat{\mathbf{x}} + \cos\phi' \hat{\mathbf{y}})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

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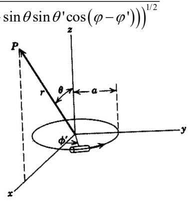
Magnetostatics example: current loop -- continued

$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\delta(\cos\theta')\delta(r'-a)(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a}\int r'^2\, dr' \, d\cos\theta' d\varphi' \frac{\sin\theta'\delta(\cos\theta')\delta(r'-a)(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})}{\left(r^2+r'^2-2rr'\left(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\right)\right)^{1/2}}$$

Completing integration over r' and θ':



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a^2}{4\pi a}\int_0^{2\pi} d\varphi' \frac{(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})}{\left(r^2+a^2-2ra\left(\sin\theta\cos(\varphi-\varphi')\right)\right)^{1/2}}$$

Let $\varphi-\varphi' \equiv \phi$

$\sin\varphi' = \sin(\varphi-\phi) = \sin\varphi\cos\phi - \cos\varphi\sin\phi$

$\cos\varphi' = \cos(\varphi-\phi) = \cos\varphi\cos\phi + \sin\varphi\sin\phi$

Remaining non-trivial terms

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})\int_0^{2\pi} d\phi \frac{\cos\phi}{\left(r^2+a^2-2ra(\sin\theta\cos\phi)\right)^{1/2}}$$

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Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})\int_0^{2\pi} d\phi \frac{\cos\phi}{\left(r^2+a^2-2ra(\sin\theta\cos\phi)\right)^{1/2}}$$

Elliptic integrals:

$$K(m) = \int_0^{\pi/2} \frac{du}{\left(1-m\sin^2 u\right)^{1/2}}$$

$$E(m) = \int_0^{\pi/2} \left(1-m\sin^2 u\right)^{1/2} du$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}4Ia\frac{(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

where: $k^2 \equiv \dfrac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$

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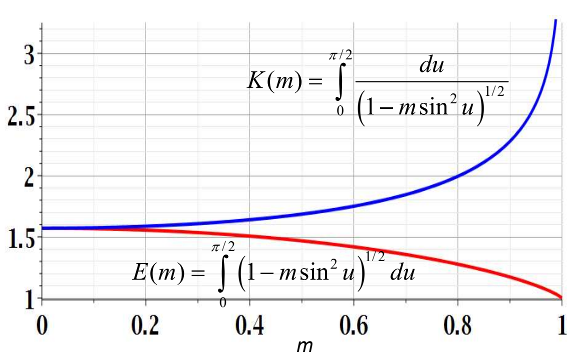

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Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}})}{(r^2+a^2+2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

where: $k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} = \frac{4ax}{x^2+z^2+a^2+2ax}$

For $\varphi = 0$: $x = r\sin\theta, \;\; y = 0, \;\; z = r\cos\theta$

$$\mathbf{A}(\mathbf{r}) = A_y(x,z)\hat{\mathbf{y}} = \frac{\mu_0}{4\pi} 4Ia\hat{\mathbf{y}} \frac{1}{(x^2+z^2+a^2+2ax)^{1/2}} \left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

where: $k^2 \equiv \frac{4ax}{x^2+z^2+a^2+2ax}$

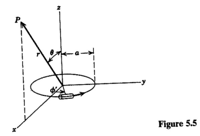
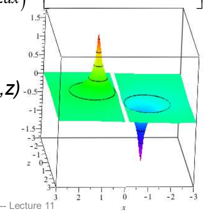


Figure 5.5

$A_y(x,z)$



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Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}})}{(r^2+a^2+2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

$$= \hat{\boldsymbol{\varphi}} \frac{\mu_0}{4\pi} \frac{4Ia}{(r^2+a^2+2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

where: $k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

Evaluation for special cases

For $k^2 \to 0$:

$$\frac{(2-k^2)K(k)-2E(k)}{k^2} \approx \frac{\pi}{16}k^2$$

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Evaluation for special case $k^2 \rightarrow 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 Ia^2}{4} \frac{r\sin\theta}{(r^2+a^2+2ra\sin\theta)^{3/2}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 Ia^2}{4} \frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2+2r^2+ar\sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2-r^2+ar\sin\theta\right)\right)}{(r^2+a^2+2ra\sin\theta)^{5/2}}$$

where $k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} \to 0$

For $\theta = 0$: $r = z$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \hat{\mathbf{z}} \frac{\mu_0 Ia^2}{2} \frac{1}{(z^2+a^2)^{3/2}}$$

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Evaluation for special case $k^2 \to 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 I a^2}{4} \frac{r\sin\theta}{\left(r^2+a^2+2ra\sin\theta\right)^{3/2}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4}\frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2+2r^2+ar\sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2-r^2+ar\sin\theta\right)\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{5/2}}$$

$$\text{where } k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} \to 0$$

For $r\to\infty$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4r^3}\left(\hat{\mathbf{r}}(2\cos\theta) + \hat{\boldsymbol{\theta}}\sin\theta\right)$$

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Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} 4Ia \frac{\left(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$$

$$\text{where: } k \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$$

Note that for spherical polar coordinates: $\hat{\boldsymbol{\varphi}} = \sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}$

$$\mathbf{A}(\mathbf{r}) = A_\phi(\mathbf{r})\hat{\boldsymbol{\varphi}}$$

$$\text{where } A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{4Ia}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\phi(\mathbf{r})\right)}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(\mathbf{r})\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

For $r\to\infty$:

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

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Other examples of current density sources:

Quantum mechanical expression for current density for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi}\left(\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r})\right)$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{2Mi}\left|R_{nl}(r)\right|^2\frac{1}{r\sin\theta}\left(Y_{lm_l}^*(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial\phi} - Y_{lm_l}(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}^*(\hat{\mathbf{r}})}{\partial\phi}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{e\hbar}{M}\frac{m_l}{r\sin\theta}\left|\Psi_{nlm_l}(\mathbf{r})\right|^2\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \dfrac{\hat{\mathbf{z}}\times\mathbf{r}}{r\sin\theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M}\frac{m_l}{r^2\sin^2\theta}\left|\Psi_{nlm_l}(\mathbf{r})\right|^2(\hat{\mathbf{z}}\times\mathbf{r})$$

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Magnetic vector potential for this case:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{J}(\mathbf{r}') = \frac{e\hbar}{M}\frac{m_l}{r'^2\sin^2\theta'}\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|(\hat{\mathbf{z}}\times\mathbf{r}')$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar m_l}{M}\int d^3r' \frac{(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|}{r'^2\sin^2\theta'}$$

For example: electron in the $nlm_l = 211$ state of H:

$$\left|\Psi_{211}(\mathbf{r}')^2\right| = \frac{1}{64\pi a^3}\left(\frac{r'}{a}\right)^2 e^{-r'/a}\sin^2\theta'$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{8\pi}\frac{e\hbar}{M}\frac{(\hat{\mathbf{z}}\times\mathbf{r})}{r^3}\left[1-e^{-r/a}\left(1+\frac{r}{a}+\frac{r^2}{2a^2}+\frac{r^3}{8a^3}\right)\right]$$

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