PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 9:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization P

C. Electric displacement D and dielectric functions

Two colloquia this week:

Colloquium: "Novel Material Platforms and Transdimensional Lattices for Metaphotonic Devices" – February 5, 2019, at 2:00 PM

Posted on January 30, 2019

Viktoriia Babicheva, PhD,
College of Optical Sciences, University of Arizona
George P. Williams, Jr. Lecture Hall, (Olin 101)
Tuesday, February 5, 2019, at 2:00 PM

Colloquium: "Light-Driven Self-Organization of Nanoparticles into Artificial Materials" Wednesday, February 6, 2019 at 4:00 PM

Posted on January 30, 2019

Zijie Yan, PhD,
Chemical & Biomolecular Engineering, Clarkson University
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, February 6, 2019, at 4:00 PM

Course schedule for Spring 2019

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/14/2019	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/23/2019
2	Wed: 01/16/2019	Chap. 1	Electrostatic energy calculations	#2	01/23/2019
3	Fri: 01/18/2019	Chap. 1	Electrostatic potentials and fields	#3	01/23/2019
	Mon: 01/21/2019	No class	Martin Luther King Holiday		
4	Wed: 01/23/2019	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/25/2019	Chap. 1 - 3	Brief introduction to numerical methods	#4	01/28/2019
6	Mon: 01/28/2019	Chap. 2 & 3	Image charge constructions	#5	01/30/2019
7	Wed: 01/30/2019	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/01/2019	Chap. 3 & 4	Spherical geometry and multipole moments	#6	02/04/2019
9	Mon: 02/04/2019	Chap. 4	Dipoles and Dielectrics	#7	02/06/2019
10	Wed: 02/06/2019				
11	Fri: 02/08/2019				
12	Mon: 02/11/2019				
13	Wed: 02/13/2019				
14	Fri: 02/15/2019				
15	Mon: 02/18/2019				

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \to \infty) = 0$ for confined charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

For $r \to \infty$: $\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}\underbrace{\int_0^\infty r'^{2+l}\,dr'\rho_{lm}(r')}_{q_{lm}}$

Notion of multipole moment:

In the spherical harmonic representation - -

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r'\, r'^l\, Y_{lm}^*(\theta',\phi')\rho(\mathbf{r}')$$

In the Cartesian representation - -

define the monopole moment q:

$$q \equiv \int d^3r'\, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r'\, \mathbf{r}'\rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} $(i,j \to x,y,z)$:

$$Q_{ij} \equiv \int d^3r' \left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\left(\int d^3r'\, r'^l\, Y_{lm}^*(\theta',\varphi')\rho(\mathbf{r}')\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi\, q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

In terms of Cartesian expansion:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2}\sum_{i,j}Q_{ij}\frac{r_i r_j}{r^5}\cdots\right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\,(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\,\delta^3(\mathbf{r})\right)$$

Poorly defined for $r \to 0$ Correct value for $r \to 0$

"Justification" of surprizing δ-function term in dipole electric field -- Assuming dipole is located at *r=0*, we need to need to evaluate the electrostatic field near *r=0*:

We will use the approximation:

$$\mathbf{E}(\mathbf{r}\approx\mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r})d^3r\right)\delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r\le R} \mathbf{E}(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

Some details:

$$\int_{r\le R} \mathbf{E}(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}} \nabla\cdot\mathcal{V}d^3r = \int_{\text{surface}}\mathcal{V}\cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = \hat{\mathbf{x}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{x}}\Phi)d^3r + \hat{\mathbf{y}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{y}}\Phi)d^3r + \hat{\mathbf{z}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{z}}\Phi)d^3r,$$

which is equal to:

$$\int_{r=R}\Phi(\mathbf{r})R^2d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R}\Phi(\mathbf{r})R^2d\Omega\hat{\mathbf{r}}.$$

Therefore --

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

More details

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r'\rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\hat{\mathbf{r}})Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases}\sin(\theta)\cos(\phi)\hat{\mathbf{x}}+\sin(\theta)\sin(\phi)\hat{\mathbf{y}}+\cos(\theta)\hat{\mathbf{z}}\\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}+Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}+Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}}\right)\end{cases}$$

$$\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = \frac{1}{3\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{r_<}{r_>^2}\sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}}')\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}+Y_{11}(\hat{\mathbf{r}}')\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}+Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}}\right)$$

$$= \frac{1}{3\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'$$

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l=1$ terms contribute, and the result of the integration reduces to:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\int d^3r'\rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'.$$

The choice of $r_<$ and $r_>$ is a choice between the integration variables r' and the sphere radius R. If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that the result is:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\frac{1}{R^2}\int d^3r'\rho(\mathbf{r}')r'\hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\varepsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_< = R$ and $r_> = r'$ and the result is:

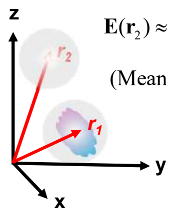
$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}R\int d^3r'\frac{\rho(\mathbf{r}')}{r'^2}\hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3}\mathbf{E}(0).$$

In summary --

Electrostatic dipolar field for dipole moment $\mathbf{p}$ at $\mathbf{r}$=0:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r})-r^2\mathbf{p}}{r^5}-\frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_2+\mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_1+\mathbf{r})$$

$$\approx \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \approx -\frac{\mathbf{p}}{3\epsilon_0}\delta(\mathbf{r}-\mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles:

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note: $\int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \int d^3r'\ \mathbf{P}(\mathbf{r}')\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$-\nabla^2\Phi(\mathbf{r}) = \nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\left(\rho_{mono}(\mathbf{r}) - \nabla\cdot\mathbf{P}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\cdot\left(\varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right) = \rho_{mono}(\mathbf{r})$$

Define Displacement field : $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law : $\nabla\cdot\mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0\chi_e\mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0(1+\chi_e)\mathbf{E}(\mathbf{r}) \equiv \varepsilon\mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

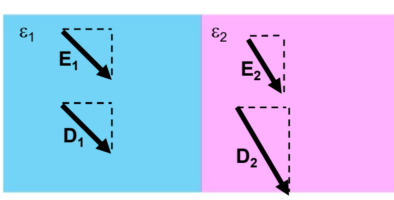
For $\rho_{\text{mono}}(\mathbf{r}) = 0$

$\nabla\cdot\mathbf{D}(\mathbf{r}) = 0$ and $\nabla\times\mathbf{E}(\mathbf{r}) = 0$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$\hat{\mathbf{r}}\cdot\mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}}\times\mathbf{E}(\mathbf{r})$

Boundary value problems in the presence of dielectrics – example:



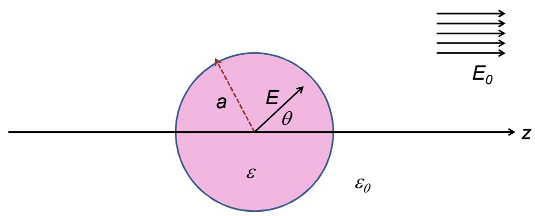
For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Boundary value problems in the presence of dielectrics – example:



$$\nabla\cdot\mathbf{D}(\mathbf{r})=0 \quad \text{and} \quad \nabla\times\mathbf{E}(\mathbf{r})=0$$

For $r\le a$ $\quad \mathbf{D}(\mathbf{r})=-\varepsilon\nabla\Phi(\mathbf{r})$

For $r>a$ $\quad \mathbf{D}(\mathbf{r})=-\varepsilon_0\nabla\Phi(\mathbf{r})$

At $r=a$:

$$\varepsilon\frac{\partial\Phi_<(\mathbf{r})}{\partial r}=\varepsilon_0\frac{\partial\Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial\Phi_<(\mathbf{r})}{\partial\theta}=\frac{\partial\Phi_>(\mathbf{r})}{\partial\theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r})=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r})=\sum_{l=0}^{\infty}\left(B_l r^l+\frac{C_l}{r^{l+1}}\right)P_l(\cos\theta)$$

At $r=a$:

$$\varepsilon\frac{\partial\Phi_<(\mathbf{r})}{\partial r}=\varepsilon_0\frac{\partial\Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial\Phi_<(\mathbf{r})}{\partial\theta}=\frac{\partial\Phi_>(\mathbf{r})}{\partial\theta}$$

For $r\to\infty$ $\quad \Phi_>(\mathbf{r})=-E_0 r\cos\theta$

Solution -- only $l=1$ contributes

$$B_1=-E_0$$

$$A_1=-\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0 \qquad C_1=\left(\frac{\varepsilon/\varepsilon_0-1}{2+\varepsilon/\varepsilon_0}\right)a^3E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r})=-\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r})=-\left(r-\left(\frac{\varepsilon/\varepsilon_0-1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0\cos\theta$$