PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 23:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

WAKE FOREST
UNIVERSITY
GRADUATE SCHOOL of
ARTS & SCIENCES

Wake Forest University's Graduate School of Arts and Sciences cordially invites you to attend the

19th Annual Graduate Student and Postdoc Research Day
Friday, March 22, 2019
Wake Forest Biotech Place
575 N. Patterson Ave

Students and postdoctoral fellows at Wake Forest University's Graduate School of Arts & Sciences will present their current research projects via poster presentations and a Three-Minute Thesis competition. Research presentations will span WFU's 30+ graduate programs including biomedical research, liberal arts, social sciences, and basic science. Wake Forest faculty, students, and the surrounding community are invited to attend and discover the Graduate School's vital role as an engine of discovery that fuels the nation's scholarly and creative enterprise.

SCHEDULE OF EVENTS
2:00 pm Poster Session
3:00 pm Three Minute Thesis Competition
4:00 pm Awards & Reception

VISITOR PARKING AND DIRECTIONS
Visitors will use P8 the visitor lot accessible via Research Parkway.
Parking Map

Take Seventh Street to Vine Street and turn left. Proceed down the hill until you reach the entrance to Biotech Place, turn right and proceed down the stairs and enter through the main building entrance.

SHUTTLE SERVICE
Wake Downtown provides direct non-stop daily shuttle service between Innovation Quarter and Wake Forest University's Reynolda Campus.
Shuttle Schedule

	Wed: 03/13/2019	No class	*Spring Break*		
	Fri: 03/15/2019	No class	*Spring Break*		
21	Mon: 03/18/2019	Chap. 9	Radiation from localized oscillating sources	#15	3/22/2019
22	Wed: 03/20/2019	Chap. 9	Radiation from oscillating sources	#16	3/25/2019
23	Fri: 03/22/2019	Chap. 9 and 10	Radiation from oscillating sources	#17	3/27/2019
24	Mon: 03/25/2019				
25	Wed: 03/27/2019				
26	Fri: 03/29/2019				
27	Mon: 04/01/2019				
28	Wed: 04/03/2019				
29	Fri: 04/05/2019				
30	Mon: 04/08/2019				
31	Wed: 04/10/2019				
32	Fri: 04/12/2019				
33	Mon: 04/15/2019				
34	Wed: 04/17/2019				
	Fri: 04/19/2019	No class	*Good Friday*		
35	Mon: 04/22/2019				
36	Wed: 04/24/2019				
	Fri: 04/26/2019		Presentations I		
	Mon: 04/29/2019		Presentations II		
	Wed: 05/01/2019		Presentations III		

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

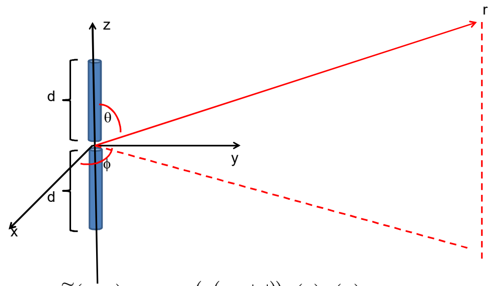
$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 <-> d



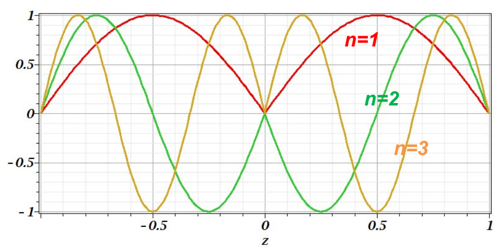
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin(k(d-|z|))\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin(k(d-|z|))\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin(k(d-|z|))\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $\quad \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz' \; e^{-ikz'\cos\theta} \;\; \sin(k(d-|z'|))$$

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, I\int_{-d}^{d} dz\, e^{-ikz\cos\theta}\ \sin\left(k\left(d-|z|\right)\right)$$

$$= \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi}\frac{e^{ikr}}{kr}\,2I\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin^2\theta}\right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

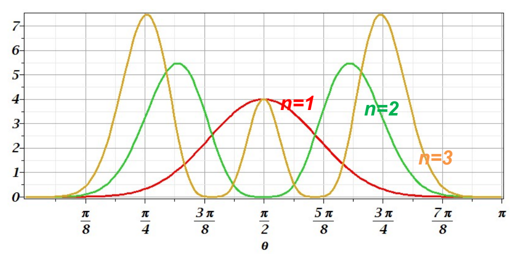
$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2c}{2\mu_0} r^2\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}}\cdot\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

Consider antenna source -- continued

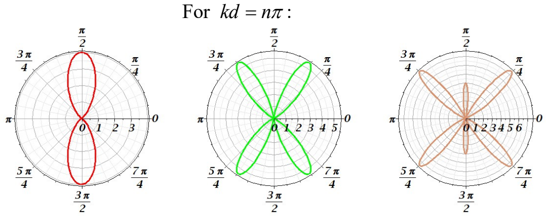
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$



Consider antenna source -- continued

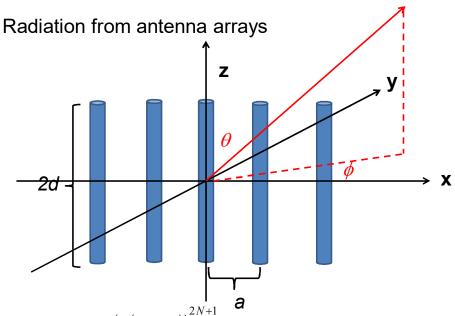
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

For $kd = n\pi$:



n=1 n=2 n=3

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin(k(d-|z|))\sum_{j=1}^{2N+1}\delta(x-(N+1-j)a)\delta(y) \quad \text{for } -d\le z\le d$$

$$k\equiv\frac{\omega}{c}=\frac{n\pi}{d}; \qquad n=1,2,3....$$

Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin(k(d-|z|))\sum_{j=1}^{2N+1}\delta(x-(N+1-j)a)\delta(y) \quad \text{for } -d\le z\le d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\,e^{-ikz\cos\theta}\,\sin(k(d-|z|))$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Radiation from antenna arrays -- continued

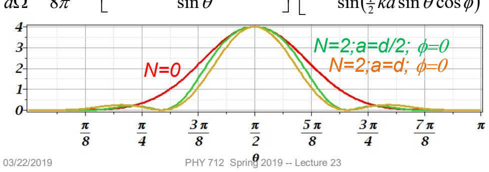
In the radiation zone :

$$\tilde{\mathbf{B}}(\mathbf{r},\omega)=\nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx ik\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)\approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

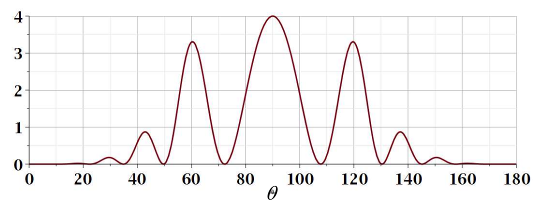
$$\frac{dP}{d\Omega}=\frac{1}{2\mu_0}r^2\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)=\frac{k^2cr^2}{2\mu_0}\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2-\left|\hat{\mathbf{r}}\cdot\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega}=\frac{\mu_0c}{8\pi^2}I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2\left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



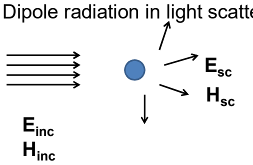
$$\frac{dP}{d\Omega}=\frac{\mu_0 c}{8\pi^2}I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2\left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}\right]^2$$

Example for $\phi=0, N=10,\ \ kd=\pi=2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Dipole radiation in light scattering by small (dielectric) particles

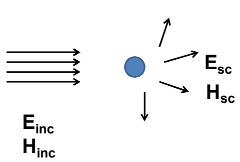


$$\mathbf{E}_{\rm inc}=\hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}} \qquad \mathbf{H}_{\rm inc}=\frac{1}{\mu_0 c}\hat{\mathbf{k}}_0\times\mathbf{E}_{\rm inc}$$

In electric dipole approximation :

$$\mathbf{E}_{\rm sc}=\frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left((\hat{\mathbf{r}}\times\mathbf{p})\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\rm sc}=\frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\rm sc}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\rm inc}=\hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}}$$

$$\mathbf{H}_{\rm inc}=\frac{1}{\mu_0 c}\hat{\mathbf{k}}_0\times\mathbf{E}_{\rm inc}$$

In electric dipole approximation:

$$\mathbf{E}_{\rm sc}=\frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left((\hat{\mathbf{r}}\times\mathbf{p})\times\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\rm sc}=\frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\rm sc}$$

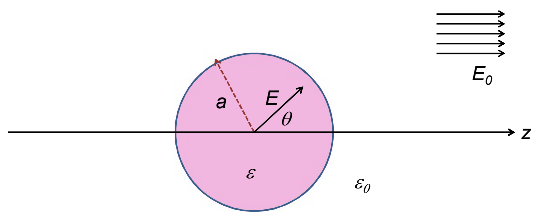
Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}_{sc}\rangle_{avg}}{\hat{\mathbf{k}}_0\cdot\langle\mathbf{S}_{inc}\rangle_{avg}}$$

$$=\frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2}=\frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r=a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\text{At } r=a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

$$\text{For } r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$$

Solution -- only $l=1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2+\varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0} \right) a^3 E_0$$

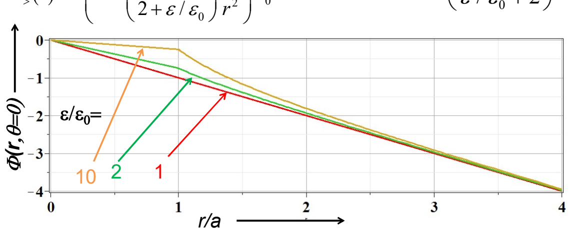
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0} \right) E_0 \, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0} \right) \frac{a^3}{r^2} \right) E_0 \cos\theta$$

Induced dipole moment:

$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

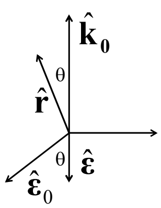
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

Scattering by dielectric sphere with permittivity ε and radius *a*:

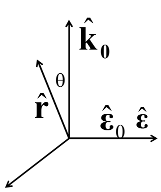


For $\mathbf{E}_{inc}$ polarized in scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) == k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

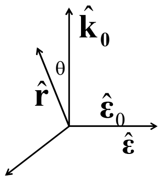
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2$$

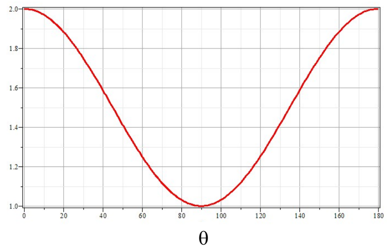
Assuming both polarizations are equally likely, average cross section is given by :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4 a^6}{2} \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left(\cos^2\theta + 1\right)$$

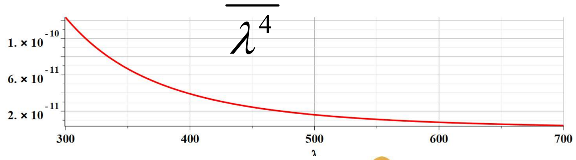
Scattering by dielectric sphere with permittivity ε and radius *a*:

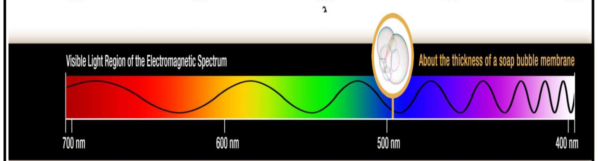


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4 a^6}{2}\left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left(\cos^2\theta + 1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of **E** and **H** fields with time dependence $e^{-i\omega t}$:

$$\nabla\times\mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla\times\mathbf{H} = -ik\mathbf{E}/Z_0$$

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{H} = 0$$

where $k \equiv \omega/c$ and $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla\times\mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla\times\mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right)=0 \qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right)=0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r}\cdot\mathbf{L}=0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r}\cdot\mathbf{H}^M_{lm} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r}\cdot\mathbf{E}^M_{lm} = 0$$

$$\mathbf{L}\cdot\mathbf{E}^M_{lm} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r}\cdot\mathbf{E}^E_{lm} \equiv -Z_0\frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r}\cdot\mathbf{H}^E_{lm} = 0$$

$$\mathbf{L}\cdot\mathbf{H}^E_{lm} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l>0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\, \mathbf{X}_{l'm'}^*(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\, \mathbf{X}_{l'm'}^*(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a^E_{lm} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a^M_{lm}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a^E_{lm}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a^M_{lm} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega}=\frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}}+a^M_{lm}\mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

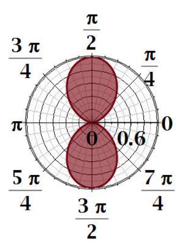
For a pure multipole radiation with either a^E_{lm} or a^M_{lm} :

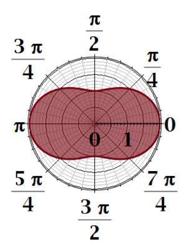
$$\frac{dP}{d\Omega}=\frac{Z_0}{2k^2}\left|a_{lm}\right|^2\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2=\frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2+(l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2+(l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l=1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2=\frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2=\left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2=\frac{3}{16\pi}\left(1+\cos^2\theta\right)$$





For example: $l=2$

$$\left|\mathbf{X}_{20}(\theta,\phi)\right|^2=\frac{15}{8\pi}\sin^2\theta\cos^2\theta \qquad \left|\mathbf{X}_{21}(\theta,\phi)\right|^2=\frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad \left|\mathbf{X}_{22}(\theta,\phi)\right|^2=\frac{5}{16\pi}\left(1-\cos^4\theta\right)$$