



Review:

## Maxwell's equations

Microscopic or vacuum form ( $\mathbf{P} = 0$ ;  $\mathbf{M} = 0$ ):

Coulomb's law :  $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law :  $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law :  $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles :  $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Review:

Formulation of Maxwell's equations in terms of vector and scalar potentials:

Lorentz gauge form -- require :  $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Note that the Lorentz gauge is consistent with the source continuity condition :  $\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0$

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Review:

Electromagnetic waves from time harmonic sources

Charge density :  $\rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega)e^{-i\omega t})$

Current density :  $\mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega)e^{-i\omega t})$

$\Rightarrow$  Scalar potential :  $\Phi(\mathbf{r}, t) = \Re(\tilde{\Phi}(\mathbf{r}, \omega)e^{-i\omega t})$

$\Rightarrow$  Vector potential :  $\mathbf{A}(\mathbf{r}, t) = \Re(\tilde{\mathbf{A}}(\mathbf{r}, \omega)e^{-i\omega t})$

For  $k \equiv \frac{\omega}{c}$ :

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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Review:

Electromagnetic waves from time harmonic sources – continued:  
Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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Review:

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \quad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \quad h_1(x) = -\left(1 + \frac{i}{x}\right) \frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin(x) - \frac{3\cos(x)}{x^2} \quad h_2(x) = i \left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x \ll 1 \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x \gg 1 \Rightarrow h_l(x) \approx (-i)^{l+1} \frac{e^{ix}}{x}$$

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Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos\theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$\tilde{\mathbf{A}}(r, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \left( \frac{e^{ikr}}{kr} \int_0^r r' dr' e^{-r'/R} \sin(kr') + \frac{\sin(kr)}{kr} \int_r^\infty r' dr' e^{-r'/R + ikr'} \right)$$

$$\approx_{r \gg R} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{kr} \frac{2Rk^2 R^2}{(k^2 R^2 + 1)^2}$$

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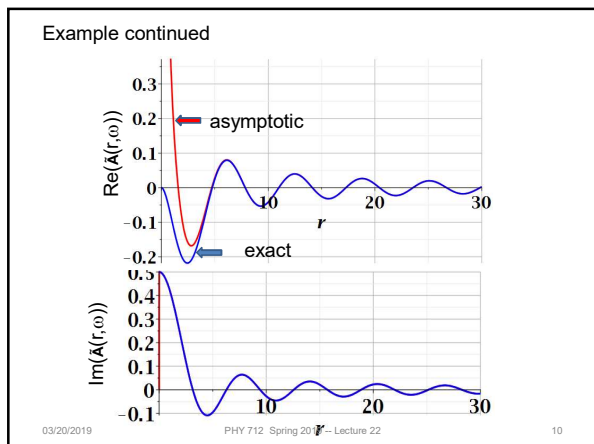
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Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case :

Define dipole moment at frequency  $\omega$  :

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

For fields outside extent of source and  $kr' \ll 1$  within the source :

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \frac{\mathbf{p}(\omega) e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i}{4\pi\omega\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

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Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\nabla\tilde{\Phi}(\mathbf{r}, \omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left( k^2 \left( (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right) + \left( \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0 c} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr}\right)$$

Power radiated for  $kr \gg 1$  :

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{\text{avg}} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega))$$

$$= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$

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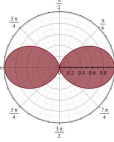
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Properties of dipole radiation field for  $kr \gg 1$ :

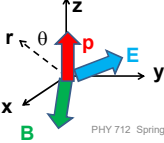
$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}))$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega))$$

Power radiated for  $kr \gg 1$ :

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{\text{avg}} = \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$


Example:



Note that vectors  $\mathbf{r}$ ,  $\mathbf{E}$ ,  $\mathbf{B}$  are mutually orthogonal

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Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For  $r \gg r'$ :  $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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For our example:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

For  $r \gg r'$ :  $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

→ Results equivalent to Bessel function expansion in the limit  $kr \rightarrow \infty$ .

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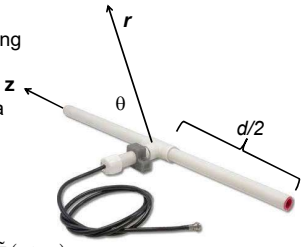
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Other radiation sources using "alternative approach"

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}', \omega) = I_0 \sin\left(\frac{kd}{2} - k|z'|\right) \delta(x)\delta(y)\hat{\mathbf{z}}$$

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Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz' e^{-ik\cos(\theta)z'} \sin\left(\frac{kd}{2} - k|z'|\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left( \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta} \right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

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Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

for  $kd = \pi$  :

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

for  $kd = 2\pi$  :

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

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Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

$kd=\pi$        $2\pi$        $3\pi$        $4\pi$        $5\pi$

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Dipole radiation in light scattering by small (dielectric) particles

$$\mathbf{E}_{inc} = \hat{\mathbf{e}}_0 E_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}} \quad \mathbf{H}_{inc} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{inc}$$

In electric dipole approximation :

$$\mathbf{E}_{sc} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \quad \mathbf{H}_{sc} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{sc}$$

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Dipole radiation in light scattering by small (dielectric) particles

Scattering cross section :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{e}}; \hat{\mathbf{k}}_0, \hat{\mathbf{e}}_0) = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{sc} \rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{inc} \rangle_{avg}}$$

$$= \frac{r^2 |\hat{\mathbf{e}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{e}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{e}} \cdot \mathbf{p}|^2$$

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Estimation of scattering dipole moment:  
 Suppose the scattering particle is a dielectric sphere with permittivity  $\epsilon$  and radius  $a$ :

$$\mathbf{p} = 4\pi a^3 \epsilon_0 \left( \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right) \mathbf{E}_{inc} \quad \mathbf{E}_{inc} = \hat{\mathbf{\epsilon}}_0 E_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}}$$

Scattering cross section :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = \frac{r^2 |\hat{\mathbf{\epsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{\epsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{\epsilon}} \cdot \mathbf{p}|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

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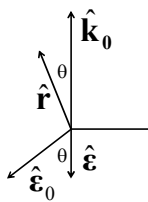
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Scattering by dielectric sphere with permittivity  $\epsilon$  and radius  $a$ :  
 For  $\mathbf{E}_{inc}$  polarized in scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 \cos^2 \theta$$

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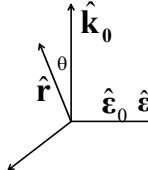
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Scattering by dielectric sphere with permittivity  $\epsilon$  and radius  $a$ :  
 For  $\mathbf{E}_{inc}$  polarized perpendicular to scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2$$

Assuming both polarizations are equally likely, average cross section is given by :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

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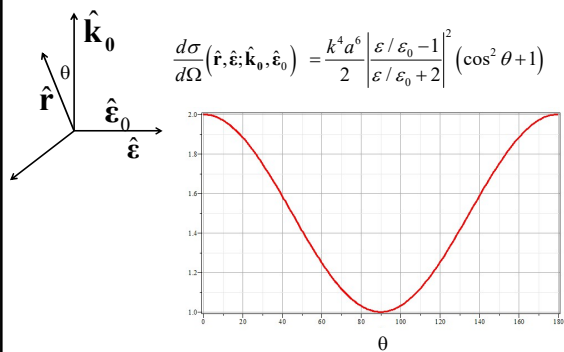
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Scattering by dielectric sphere with permittivity  $\epsilon$  and radius  $a$ :



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