

**PHY 712 Electrodynamics**  
**9-9:50 AM Olin 105**

**Plan for Lecture 17:**

**Continue reading Chapter 7**

- 1. Real and imaginary contributions to electromagnetic response**
- 2. Frequency dependence of dielectric materials; Drude model**
- 3. Kramers-Kronig relationships**

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13	Wed: 02/13/2019	Chap. 5	Magnetic dipoles and dipolar fields	#10	02/15/2019
14	Fri: 02/15/2019	Chap. 6	Maxwell's Equations	#11	02/18/2019
15	Mon: 02/18/2019	Chap. 6	Electromagnetic energy and forces	#12	02/20/2019
16	Wed: 02/20/2019	Chap. 7	Electromagnetic plane waves	#13	02/22/2019
17	Fri: 02/22/2019	Chap. 7	Electromagnetic plane waves	#14	02/25/2019
18	Mon: 02/25/2019				
19	Wed: 02/27/2019				
20	Fri: 03/01/2019				
	Mon: 03/04/2019	No class	APS March Meeting	Take Home Exam	
	Wed: 03/06/2019	No class	APS March Meeting	Take Home Exam	
	Fri: 03/08/2019	No class	APS March Meeting	Take Home Exam	
	Mon: 03/11/2019	No class	Spring Break		
	Wed: 03/13/2019	No class	Spring Break		
	Fri: 03/15/2019	No class	Spring Break		
21	Mon: 03/18/2019				
22	Wed: 03/20/2019				

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## Maxwell's equations

For linear isotropic media and no sources:  $\mathbf{D} = \epsilon\mathbf{E}$ ;  $\mathbf{B} = \mu\mathbf{H}$

Coulomb's law:  $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law:  $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law:  $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles:  $\nabla \cdot \mathbf{B} = 0$

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Plane wave solutions to sourceless Maxwell's equations;  
extension of analysis to complex dielectric functions

For simplicity assume that  $\mu = \mu_0$

Suppose the dielectric function is complex:

$$\epsilon = \epsilon_R + i\epsilon_I \quad \frac{\epsilon}{\epsilon_0} = (n_R + in_I)^2 \equiv \alpha + i\beta$$

$$n_R^2 - n_I^2 = \alpha \quad 2n_R n_I = \beta$$

$$n_I = \sqrt{n_R^2 - \alpha} \quad 4n_R^4 - 4n_R^2 \alpha - \beta^2 = 0$$

$$n_R = \left( \frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2} \right)^{1/2} \quad n_I = \left( \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2} \right)^{1/2}$$

$$\mathbf{E}(\mathbf{r}, t) = \Re \left( \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - ct} \right) = \Re \left( \mathbf{E}_0 e^{i\frac{\omega}{c}(\mathbf{n}_R \mathbf{k} \cdot \mathbf{r} - ct)} \right) e^{-\frac{\omega}{c} \mathbf{n}_I \cdot \mathbf{r}}$$

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Paul Karl Ludwig Drude 1863-1906



Scanned at the American Institute of Physics

[http://photos.aip.org/history/Thumbnails/drude\\_paul\\_a1.jpg](http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg)

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Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \dot{\delta \mathbf{r}}$$

$$\text{For } \delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}, \quad \delta \mathbf{r}_0 = \frac{q\mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Induced dipole:

$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

Displacement field:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

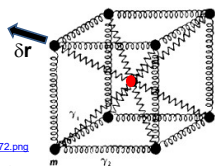
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

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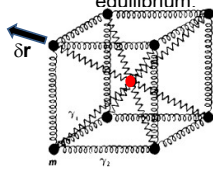
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Drude model:  
Vibration of particle of charge  $q$  and mass  $m$  near equilibrium:



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$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$  represents dissipation of energy.
- $\omega_0$  represents the natural frequency of the vibration;  $\omega_0=0$  would represent a free (unbound) particle

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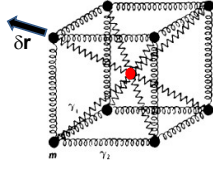
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$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

For  $\delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}$ ,  $\delta \mathbf{r}_0 = \frac{q\mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

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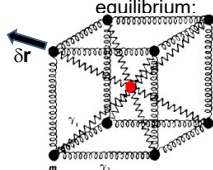
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Drude model:  
Vibration of particle of charge  $q$  and mass  $m$  near equilibrium:



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$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N \sum_i f_i \mathbf{p}_i$$

$N \equiv$  number of dipoles/volume  
 $f_i \equiv$  fraction of type  $i$  dipoles

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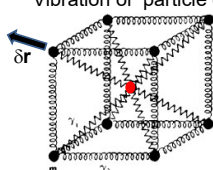
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Drude model:  
 Vibration of particle of charge  $q$  and mass  $m$  near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + N \sum_i f_i \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \delta \mathbf{r} = \frac{q_i^2 \mathbf{E}_0}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} e^{-i\omega t}$$

$$\epsilon \mathbf{E} = \epsilon_0 \mathbf{E}_0 e^{-i\omega t} \left( 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \right)$$

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Drude model dielectric function:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\epsilon_R(\omega)}{\epsilon_0} + i \frac{\epsilon_I(\omega)}{\epsilon_0}$$

$$\frac{\epsilon_R(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

$$\frac{\epsilon_I(\omega)}{\epsilon_0} = N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{\omega \gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

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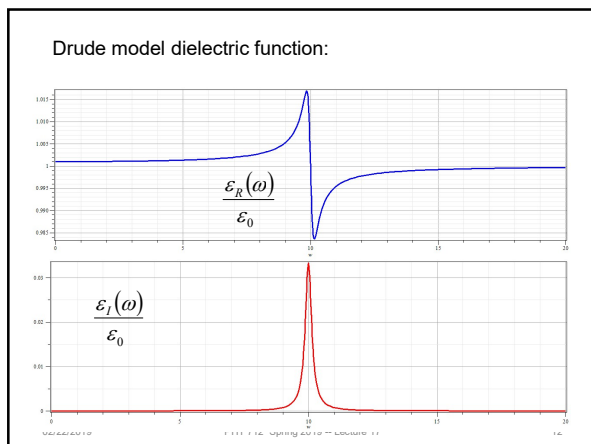
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Drude model dielectric function – some analytic properties:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For  $\omega \gg \omega_i$

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{1}{\omega^2} \left( N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \right)$$

$$\equiv 1 - \frac{\omega_p^2}{\omega^2}$$

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Drude model dielectric function – some analytic properties:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For  $\omega_0 = 0$  (representing a free particle of charge  $q_0$ , mass  $m_0$ )

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_{i>0} f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^2}{\epsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)}$$

$$\equiv \frac{\epsilon_b(\omega)}{\epsilon_0} + i \frac{\sigma(\omega)}{\epsilon_0 \omega}$$

Some details:

$$\mathbf{D} = \epsilon_b \mathbf{E} \quad \mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\epsilon_b) \mathbf{E} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = -i\omega \left( \epsilon_b + \frac{i\sigma}{\omega} \right) \mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^2}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

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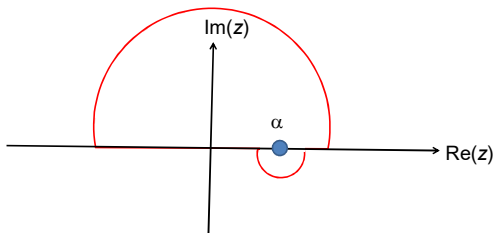
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Analytic properties of the dielectric function (in the Drude model or from “first principles” -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function  $f(z)$ :

$$\oint dz f(z) = 0 \quad f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z - \alpha}$$



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Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z - \alpha} = \frac{1}{2\pi i} \left( \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \int_{\infty}^{-\infty} dz \frac{f(z)}{z - \alpha} \right)$$

=0

$$f(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

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Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Suppose  $f(z_R) = f_R(z_R) + if_I(z_R)$ :

$$\Rightarrow \frac{1}{2} (f_R(\alpha) + if_I(\alpha)) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R) + if_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

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Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

This Kramers - Kronig transform is useful for the dielectric function  
 when  $f(z_R) \Rightarrow \frac{\epsilon(\omega)}{\epsilon_0} - 1$

Must show that: 1.  $f(z)$  is analytic for  $z_I \geq 0$   
 2.  $f(z)$  vanishes for  $z \rightarrow \infty$

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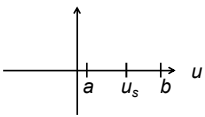
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Some practical considerations



Principal parts integration :

$$P \int_a^b du g(u) = \lim_{\nu \rightarrow 0} \left( \int_a^{u_s-\nu} du g(u) + \int_{u_s+\nu}^b du g(u) \right)$$

Example :

$$P \int_a^b du \frac{1}{u-u_s} = \lim_{\nu \rightarrow 0} \left( \int_a^{u_s-\nu} du \frac{1}{u-u_s} + \int_{u_s+\nu}^b du \frac{1}{u-u_s} \right)$$

$$= \lim_{\nu \rightarrow 0} \left( \ln \left( \frac{\nu}{u_s-a} \right) + \ln \left( \frac{b-u_s}{\nu} \right) \right) = \ln \left( \frac{b-u_s}{u_s-a} \right)$$

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More practical considerations

For dielectric function  $\epsilon(\omega)$ :

$$\epsilon(-\omega) = \epsilon^*(\omega)$$

$$\Rightarrow \epsilon_R(-\omega) = \epsilon_R(\omega)$$

$$\Rightarrow \epsilon_I(-\omega) = -\epsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of  $f(z) \Rightarrow \frac{\epsilon(z)}{\epsilon_0} - 1$

Must show that :

- $f(z)$  is analytic for  $z_i \geq 0$
- $f(z)$  vanishes for  $z \rightarrow \infty$  (for  $z_i \geq 0$ )

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Analysis for Drude model dielectric function:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let  $f(z) = \frac{\epsilon(z)}{\epsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For  $|z| \gg \omega_i$

$$f(z) \approx -\frac{1}{z^2} \left( N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \right) \Rightarrow \text{vanishes at large } z$$

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Analysis for Drude model dielectric function – continued --  
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$$f(z) \text{ has poles } z_p \text{ at } \omega_i^2 - z_p^2 - iz_p\gamma_i = 0$$

$$z_p = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that  $\Im(z_p) \leq 0 \Rightarrow f(z)$  is analytic for  $\Im(z_p) > 0$

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Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_r(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_i(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_i(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left( \frac{\varepsilon_r(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with } \varepsilon_r(-\omega) = \varepsilon_r(\omega); \quad \varepsilon_i(-\omega) = -\varepsilon_i(\omega)$$

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Further comments on analytic behavior of dielectric function

"Causal" relationship between  $\mathbf{E}$  and  $\mathbf{D}$  fields:

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r}, t) + \int_0^{\infty} d\tau G(\tau) \mathbf{E}(\mathbf{r}, t - \tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau G(\tau) e^{i\omega\tau}$$

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