

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 33:
Review – radiating systems

- **Coments/corrections on Cherenkov radiation**
- **Solution of Maxwell's equations with sources**
- **Liénard-Wiechert potentials**
- **Time periodic sources**

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24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018	Chap. 14	Radiation from accelerated particles		
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018	Chap. 14	Synchrotron radiation	#19	04/06/2018
30	Wed: 04/04/2018	Chap. 14	Synchrotron radiation	#20	04/09/2018
31	Fri: 04/06/2018	Chap. 15	Radiation from collisions of charged particles		
32	Mon: 04/09/2018	Chap. 13	Cherenkov radiation		
33	Wed: 04/11/2018		Review		
34	Fri: 04/13/2018		Review		
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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Events

Colloquium: "A Multi-Purposed Observational Approach to Measuring Exoplanet Magnetic Fields," April 11, 2018, 4pm

P. Wilson Casary, PhD, School of Earth and Space Exploration, Arizona State University
 George P. Williams, Jr. Lecture Hall, (Olin 101) Westside, April 11, 2018, at 4:00 PM
 (more info)

Colloquium: "When Galaxies Merge, What Happens to Their Supermassive Black Holes??" Apr. 18, 2018, at 4 PM

Dr. Tamara Bogdanovic – Associate Professor in the Center for Relativistic Astrophysics, School of Physics, Georgia Tech
 George P. Williams, Jr. Lecture Hall, (Olin 101) Westside, April 18, 2018, at...

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Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

$$\nabla^2 \Phi - \mu\epsilon \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{4\pi}{\epsilon} \rho$$

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c} \mathbf{J}$$

Source: charged particle moving on trajectory $\mathbf{R}_q(t)$:

$$\rho(\mathbf{r}, t) = q \delta(\mathbf{r} - \mathbf{R}_q(t))$$

$$\mathbf{J}(\mathbf{r}, t) = q \dot{\mathbf{R}}_q(t) \delta(\mathbf{r} - \mathbf{R}_q(t)) \quad q$$



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Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r}, t) = \frac{q}{\epsilon} \frac{1}{|R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r)|}$$

$$\mathbf{A}(\mathbf{r}, t) = q\mu \frac{\boldsymbol{\beta}_n}{|R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r)|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \quad c_n \equiv \sqrt{\mu\epsilon} c \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

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Consider a particle moving at constant velocity \mathbf{v} ; $v > c_n$

Some algebra

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

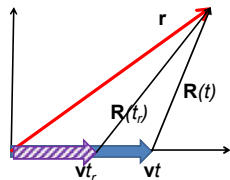
$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r) = |\mathbf{R}(t) + \mathbf{v}(t - t_r)|$$

Quadratic equation for $(t - t_r)c_n$:

$$((t - t_r)c_n)^2 = R^2(t) + 2\mathbf{R}(t) \cdot \boldsymbol{\beta}_n (t - t_r)c_n + \beta_n^2 ((t - t_r)c_n)^2$$

$$(t - t_r)c_n = \frac{-\mathbf{R}(t) \cdot \boldsymbol{\beta}_n \pm \sqrt{(\mathbf{R}(t) \cdot \boldsymbol{\beta}_n)^2 - (\beta_n^2 - 1)R^2(t)}}{\beta_n^2 - 1}$$



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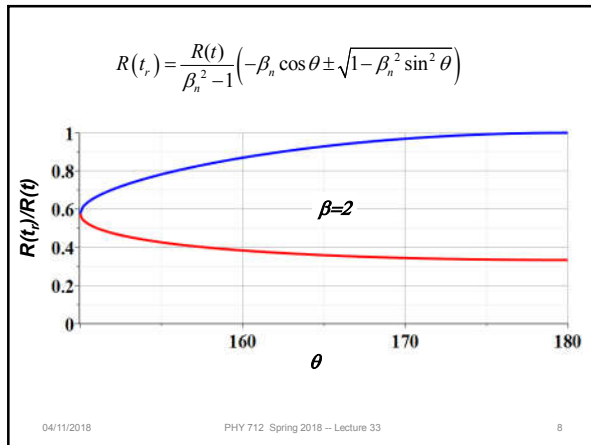
$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$
 $(t - t_r)c_n = R(t_r)$
 $R(t_r) - \mathbf{R}(t_r) \cdot \hat{\mathbf{p}}_n =$
 $(t - t_r)c_n(1 - \beta_n^2) - \mathbf{R}(t) \cdot \hat{\mathbf{p}}_n$
 $= R(t_r)(1 - \beta_n^2) - \mathbf{R}(t) \cdot \hat{\mathbf{p}}_n$

$$R(t_r) = \frac{-\mathbf{R}(t) \cdot \hat{\mathbf{p}}_n \pm \sqrt{(\mathbf{R}(t) \cdot \hat{\mathbf{p}}_n)^2 - (\beta_n^2 - 1)R^2(t)}}{\beta_n^2 - 1}$$

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1} \left(-\beta_n \cos \theta \pm \sqrt{1 - \beta_n^2 \sin^2 \theta} \right) = (t - t_r)c_n$$

$$R(t_r) - \mathbf{R}(t_r) \cdot \hat{\mathbf{p}}_n = \mp R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}$$

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Liénard-Wiechert potentials for two different retarded times:

$$\Phi(\mathbf{r}, t) = \frac{q}{\epsilon} \frac{1}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}} \Big|_{t_r}$$

$$\mathbf{A}(\mathbf{r}, t) = q\mu \frac{\hat{\mathbf{p}}_n}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}} \Big|_{t_r}$$

For $\beta_n > 1$, the range of θ is limited:

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1} \left(-\beta_n \cos \theta \pm \sqrt{1 - \beta_n^2 \sin^2 \theta} \right) \geq 0$$

$\Rightarrow \theta \leq \sin^{-1}(1/\beta_n) \equiv \theta_c$ and $\pi \geq \theta_c \geq \pi/2$ **→ Diagram is not correct!**

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Physical fields for $\beta_n > 1$ -- two retarded solutions contribute

$\theta \leq \sin^{-1}\left(\frac{1}{\beta_n}\right)$
 Define $\cos\theta_c \equiv -\sqrt{1-\frac{1}{\beta_n^2}}$
 $\Rightarrow \cos\theta \leq \cos\theta_c$

Adding two solutions; in terms of Heaviside $\Theta(x)$:

$$\Phi(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{1}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos\theta_c - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r}, t) = 2q\mu \frac{\beta_n}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos\theta_c - \cos\theta(t))$$

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Physical fields for $\beta > 1$

$$\Phi(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{1}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos\theta_c - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r}, t) = 2q\mu \frac{\beta_n}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos\theta_c - \cos\theta(t))$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi - \frac{1}{c_n} \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1-\beta_n^2 \sin^2 \theta}} \times \left(\frac{\beta_n^2 - 1}{1-\beta_n^2 \sin^2 \theta} \Theta(\cos\theta_c - \cos\theta(t)) + \sqrt{\beta_n^2 - 1} \delta(\cos\theta_c - \cos\theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r}, t) = -\beta_n \sin\theta (\hat{\theta} \times \mathbf{E}(\mathbf{r}, t))$$

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Intermediate steps:

$$\frac{d\theta}{dt} = \frac{v \sin\theta}{R} \quad \frac{dR}{dt} = -v \cos\theta$$

Using instantaneous polar coordinates: $\nabla \equiv \hat{\mathbf{R}} \frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial}{\partial \theta}$

$$\nabla \Theta(\cos\theta_c - \cos\theta(t)) = \delta(\cos\theta_c - \cos\theta(t)) \frac{\sin\theta(t)}{R(t)} \hat{\boldsymbol{\theta}}$$

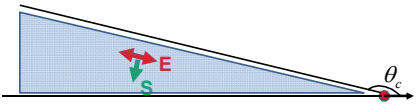
$$\frac{\partial \Theta(\cos\theta_c - \cos\theta(t))}{\partial t} = \delta(\cos\theta_c - \cos\theta(t)) \frac{v \sin^2 \theta(t)}{R(t)}$$

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Cherenkov radiation observed near the angle θ_c -- continued

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1 - \beta_n^2 \sin^2 \theta}} \times$$

$$\left(\frac{\beta_n^2 - 1}{1 - \beta_n^2 \sin^2 \theta} \Theta(\cos \theta_c - \cos \theta(t)) + \sqrt{\beta_n^2 - 1} \delta(\cos \theta_c - \cos \theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r}, t) = -\beta_n \sin \theta (\hat{\theta} \times \mathbf{E}(\mathbf{r}, t))$$


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General review --

Maxwell's equations

SI units; microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$;
 $\mu = \mu_0$; $\epsilon = \epsilon_0$):

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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General review --

Maxwell's equations

Gaussian units; microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$;
 $\epsilon = 1$; $\mu = 1$):

Coulomb's law: $\nabla \cdot \mathbf{E} = 4\pi\rho$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Keep SI units for the moment --

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

or $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

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Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2 \Phi_L - \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = -\rho / \epsilon_0$$

$$\nabla^2 \mathbf{A}_L - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation :

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r}, t) \Rightarrow$ wave field $f(\mathbf{r}, t) \Rightarrow$ source

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2 \Psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Psi(\mathbf{r}, t)}{\partial t^2} = -4\pi f(\mathbf{r}, t)$$

Green's function :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Formal solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' G(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

For the case of isotropic boundary values at infinity :

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right) \right)$$

Formal solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right) \right) f(\mathbf{r}', t')$$

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Formal solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

Two types of systems:

- ⇒ Time harmonic source: $f(\mathbf{r}, t) = \tilde{f}(\mathbf{r}, \omega)e^{-i\omega t}$
 - ⇒ spherical harmonic expansion;
 - spherical Bessel functions
- ⇒ Charged particle with known trajectory: $\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{R}_q(t))$

→ Liénard-Wiechert potentials

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Liénard-Wiechert potentials

Solution for scalar potential $\Phi(\mathbf{r}, t)$

(assuming zero homogeneous solution):

$$\begin{aligned} \Phi(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) \delta(\mathbf{r}' - \mathbf{R}_q(t')) \\ &= \frac{q}{4\pi\epsilon_0} \int dt' \frac{1}{|\mathbf{r} - \mathbf{R}_q(t')|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{R}_q(t')|\right)\right) \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right| - \left(\mathbf{r} - \mathbf{R}_q(t_r)\right) \cdot \dot{\mathbf{R}}_q(t_r) / c} \quad * \\ &\equiv \frac{q}{4\pi\epsilon_0} \frac{1}{\left|R(t_r) - \mathbf{R}(t_r) \cdot \boldsymbol{\beta}(t_r)\right|} \quad * \text{ Recall that: } \end{aligned}$$

where $t_r \equiv t - R(t_r) / c$

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left|\frac{df}{dx}(x_i)\right|_{f(x_i)=0}}$$

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