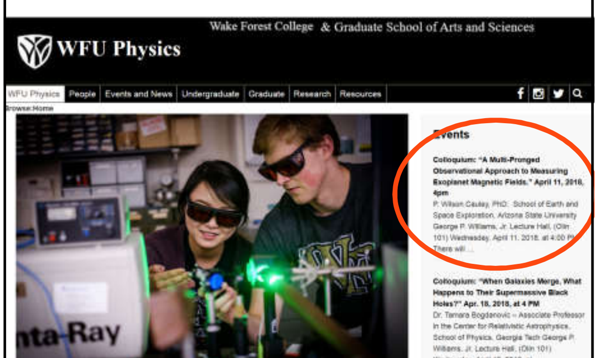
PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 33:

Review – radiating systems

- Coments/corrections on Cherenkov radiation
- Solution of Maxwell's equations with sources
- Liénard-Wiechert potentials
- Time periodic sources

24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018	Chap. 14	Radiation from accelerated particles		
	Fri: 03/30/2018	No class	*Good Friday*		
29	Mon: 04/02/2018	Chap. 14	Synchrotron radiation	#19	04/06/2018
30	Wed: 04/04/2018	Chap. 14	Synchrotron radiation	#20	04/09/2018
31	Fri: 04/06/2018	Chap. 15	Radiation from collisions of charged particles		
32	Mon: 04/09/2018	Chap. 13	Cherenkov radiation		
33	Wed: 04/11/2018		Review		
34	Fri: 04/13/2018		Review		
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		



Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

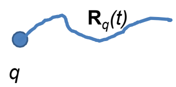
$$\nabla^2\Phi - \mu\varepsilon \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho$$

$$\nabla^2\mathbf{A} - \mu\varepsilon \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c}\mathbf{J}$$

Source: charged particle moving on trajectory $\mathbf{R}_q(t)$:

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{R}_q(t))$$

$$\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta(\mathbf{r}-\mathbf{R}_q(t))$$



Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon}\frac{1}{|R(t_r) - \boldsymbol{\beta}_n\cdot\mathbf{R}(t_r)|}$$

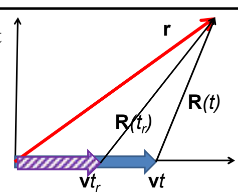
$$\mathbf{A}(\mathbf{r},t) = q\mu\frac{\boldsymbol{\beta}_n}{|R(t_r) - \boldsymbol{\beta}_n\cdot\mathbf{R}(t_r)|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \sqrt{\mu\varepsilon}\, c \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

Consider a particle moving at constant velocity $\mathbf{v}$; $v > c_n$



Some algebra

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

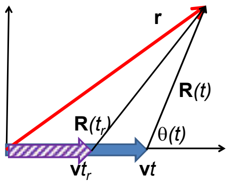
$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t-t_r)$$

$$(t-t_r)c_n = R(t_r) = |\mathbf{R}(t) + \mathbf{v}(t-t_r)|$$

Quadratic equation for $(t-t_r)c_n$:

$$((t-t_r)c_n)^2 = R^2(t) + 2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t-t_r)c_n + \beta_n^2((t-t_r)c_n)^2$$

$$(t-t_r)c_n = \frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n \pm \sqrt{(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n)^2 - (\beta_n^2-1)R^2(t)}}{\beta_n^2 - 1}$$



$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r)$$

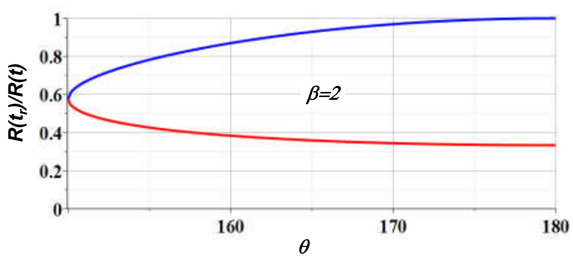
$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n = (t - t_r)c_n\left(1-\beta_n^2\right) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n = R(t_r)\left(1-\beta_n^2\right) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$R(t_r) = \frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n \pm \sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2 - \left(\beta_n^2 - 1\right)R^2(t)}}{\beta_n^2 - 1}$$

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1}\left(-\beta_n\cos\theta \pm \sqrt{1-\beta_n^2\sin^2\theta}\right) = (t - t_r)c_n$$

$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n = \mp R(t)\sqrt{1-\beta_n^2\sin^2\theta}$$

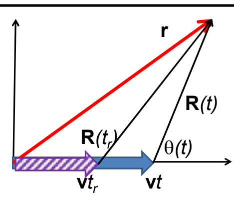
$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1}\left(-\beta_n\cos\theta \pm \sqrt{1-\beta_n^2\sin^2\theta}\right)$$



Liénard-Wiechert potentials for two different retarded times:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon}\left.\frac{1}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\right|_{t_r}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu\left.\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\right|_{t_r}$$



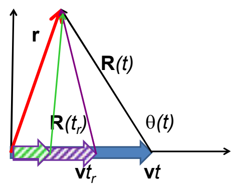
For $\beta_n > 1$, the range of θ is limited:

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1}\left(-\beta_n\cos\theta \pm \sqrt{1-\beta_n^2\sin^2\theta}\right) \geq 0$$

$$\Rightarrow \theta \leq \sin^{-1}(1/\beta_n) \equiv \theta_c \quad \text{and} \quad \pi \geq \theta_c \geq \pi/2$$

➔Diagram is not correct!

Physical fields for $\beta_n > 1$ -- two retarded solutions contribute



$$\theta \leq \sin^{-1}\left(\frac{1}{\beta_n}\right)$$

Define $\cos\theta_C \equiv -\sqrt{1-\frac{1}{\beta_n^2}}$

$$\Rightarrow \cos\theta \leq \cos\theta_C$$

Adding two solutions; in terms of Heaviside $\Theta(x)$:

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

Physical fields for $\beta > 1$

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

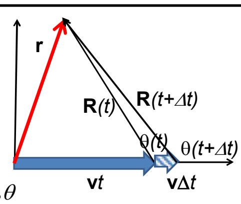
$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi - \frac{1}{c_n}\frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{\hat{\mathbf{R}}}{(R(t))^2\sqrt{1-\beta_n^2\sin^2\theta}} \times$$

$$\left(\frac{\beta_n^2-1}{1-\beta_n^2\sin^2\theta}\Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n^2-1}\;\delta(\cos\theta_C - \cos\theta(t))\right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n\sin\theta\left(\hat{\theta}\times\mathbf{E}(\mathbf{r},t)\right)$$

Intermediate steps:



$$\frac{d\theta}{dt} = \frac{v\sin\theta}{R} \qquad \frac{dR}{dt} = -v\cos\theta$$

Using instantaneous polar coordinates: $\nabla \equiv \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial\theta}$

$$\nabla\Theta\left(\cos\theta_C - \cos\theta(t)\right) = \delta\left(\cos\theta_C - \cos\theta(t)\right)\frac{\sin\theta(t)}{R(t)}\hat{\boldsymbol{\theta}}$$

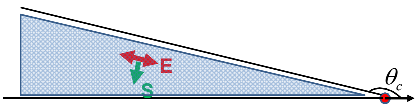
$$\frac{\partial\Theta\left(\cos\theta_C - \cos\theta(t)\right)}{\partial t} = \delta\left(\cos\theta_C - \cos\theta(t)\right)\frac{v\sin^2\theta(t)}{R(t)}$$

Cherenkov radiation observed near the angle θ_c -- continued

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{\hat{\mathbf{R}}}{(R(t))^2\sqrt{1-\beta_n^2\sin^2\theta}} \times$$

$$\left(\frac{\beta_n^2-1}{1-\beta_n^2\sin^2\theta}\Theta(\cos\theta_C-\cos\theta(t))+\sqrt{\beta_n^2-1}\;\delta(\cos\theta_c-\cos\theta(t))\right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n\sin\theta\left(\hat{\theta}\times\mathbf{E}(\mathbf{r},t)\right)$$



General review --

Maxwell's equations

SI units; microscopic or vacuum form ($\mathbf{P}=0$; $\mathbf{M}=0$; $\mu=\mu_0$; $\varepsilon=\varepsilon_0$):

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

General review --

Maxwell's equations

Gaussian units; microscopic or vacuum form ($\mathbf{P}=0$; $\mathbf{M}=0$; $\varepsilon=1;\mu=1$):

Coulomb's law: $\nabla\cdot\mathbf{E}=4\pi\rho$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Keep SI units for the moment --

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2\Phi_L - \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = -\rho/\varepsilon_0$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0\mathbf{J}$$

Consider the general form of the 3-dimensional wave equation:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r},t) \Rightarrow$ wave field $\quad f(\mathbf{r},t) \Rightarrow$ source

Solution of Maxwell's equations in the Lorentz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2} = -4\pi f(\mathbf{r},t)$$

Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Two types of systems:

$\Rightarrow$ Time harmonic source: $f(\mathbf{r},t) = \tilde{f}(r,\omega)e^{-i\omega t}$

$\Rightarrow$ spherical harmonic expansion; spherical Bessel functions

$\Rightarrow$ Charged particle with known trajectory: $\rho(\mathbf{r},t) = q\delta(\mathbf{r} - \mathbf{R}_q(t))$

➔**Liénard-Wiechert potentials**

Liénard-Wiechert potentials

Solution for scalar potential $\Phi(\mathbf{r},t)$ (assuming zero homogeneous solution):

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0} \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) \delta(\mathbf{r}' - \mathbf{R}_q(t'))$$

$$= \frac{q}{4\pi\varepsilon_0} \int dt' \frac{1}{|\mathbf{r}-\mathbf{R}_q(t')|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{R}_q(t')|\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{\left||\mathbf{r}-\mathbf{R}_q(t_r)| - (\mathbf{r}-\mathbf{R}_q(t_r)) \cdot \dot{\mathbf{R}}_q(t_r)/c\right|} \quad *$$

$$\equiv \frac{q}{4\pi\varepsilon_0} \frac{1}{|R(t_r) - \mathbf{R}(t_r) \cdot \boldsymbol{\beta}(t_r)|}$$

where $t_r \equiv t - R(t_r)/c$

* Recall that:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left|\frac{df}{dx}(x_i)\right|_{f(x_i)=0}}$$