

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 31:
Start reading Chap. 15 –
Radiation from collisions of charged particles

- 1. Overview**
- 2. X-ray tube**
- 3. Radiation from Rutherford scattering**
- 4. Other collision models**

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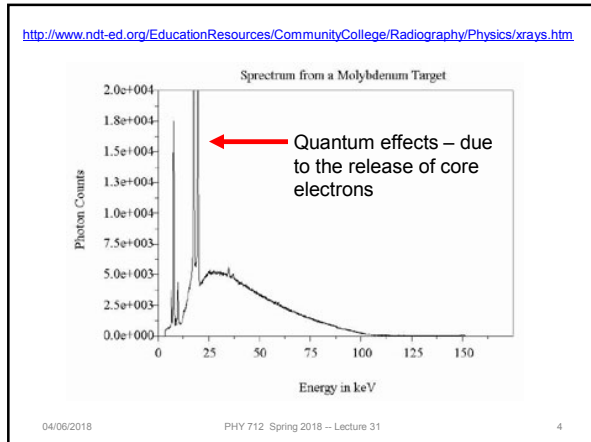
23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018	Chap. 14	Radiation from accelerated particles		
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018	Chap. 14	Synchrotron radiation	#19	04/06/2018
30	Wed: 04/04/2018	Chap. 14	Synchrotron radiation	#20	04/09/2018
31	Fri: 04/06/2018	Chap. 15	Radiation from collisions of charged particles		
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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Generation of X-rays in a Coolidge tube
<https://www.orau.org/ptp/collection/xraytubescoolidge/coolidgeinformation.htm>

The diagram illustrates the internal structure of a Coolidge tube. It consists of a central bulb with two arms extending from it: an Anode Arm on the left and a Cathode Arm on the right. At the end of the Cathode Arm is a Cathode, which emits an Electron Beam towards a Tungsten Anode located at the end of the Anode Arm. The point of impact between the electron beam and the anode is labeled as the source of the X-Ray Beam, which is shown as a fan of rays emanating from that point.

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Radiation during collisions

Intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt e^{i\omega(t - \hat{r} \cdot \mathbf{R}_s(t)/c)} \frac{d}{dt} \left[\frac{\hat{r} \times (\hat{r} \times \boldsymbol{\beta})}{1 - \hat{r} \cdot \boldsymbol{\beta}} \right] \right|^2$$

Note that $\hat{r} \times (\hat{r} \times \boldsymbol{\beta}) = (\boldsymbol{\epsilon}_\parallel \cdot \boldsymbol{\beta}) \boldsymbol{\epsilon}_\parallel + (\boldsymbol{\epsilon}_\perp \cdot \boldsymbol{\beta}) \boldsymbol{\epsilon}_\perp$

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\epsilon}$ and frequency $\omega \rightarrow 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\epsilon} \cdot \left(\frac{\boldsymbol{\beta}(t + \tau)}{1 - \hat{r} \cdot \boldsymbol{\beta}(t + \tau)} - \frac{\boldsymbol{\beta}(t)}{1 - \hat{r} \cdot \boldsymbol{\beta}(t)} \right) \right|^2$$

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Radiation during collisions -- continued

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\epsilon}$ and frequency $\omega \rightarrow 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\epsilon} \cdot \left(\frac{\boldsymbol{\beta}(t + \tau)}{1 - \hat{r} \cdot \boldsymbol{\beta}(t + \tau)} - \frac{\boldsymbol{\beta}(t)}{1 - \hat{r} \cdot \boldsymbol{\beta}(t)} \right) \right|^2$$

Non-relativistic limit:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\epsilon} \cdot (\Delta \boldsymbol{\beta}) \right|^2 \quad \Delta \boldsymbol{\beta} \equiv \boldsymbol{\beta}(t + \tau) - \boldsymbol{\beta}(t)$$

Relativistic collision with small $|\Delta \boldsymbol{\beta}| \equiv \boldsymbol{\beta}(t + \tau) - \boldsymbol{\beta}(t)$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\epsilon} \cdot \left(\frac{\Delta \boldsymbol{\beta} + \hat{r} \times (\boldsymbol{\beta} \times \Delta \boldsymbol{\beta})}{(1 - \hat{r} \cdot \boldsymbol{\beta})^2} \right) \right|^2$$

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Radiation during collisions -- continued

Relativistic collision with small $|\Delta\beta|$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \mathbf{e} \cdot \left(\frac{\Delta\beta + \hat{\mathbf{r}} \times (\beta \times \Delta\beta)}{(1 - \hat{\mathbf{r}} \cdot \beta)^2} \right) \right|^2$$

Also assume $\Delta\beta$ is perpendicular to $\mathbf{r} - \beta$ plane

Expressions (averaging over ϕ) for \parallel or \perp polarization:

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\beta|^2 \frac{(\beta - \cos\theta)^2}{(1 - \beta \cos\theta)^4} \text{ polarization in } r \text{ and } \beta \text{ plane}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\beta|^2 \frac{1}{(1 - \beta \cos\theta)^2} \text{ polarization perpendicular to } r \text{ and } \beta \text{ plane}$$

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Some details:

$$\hat{\mathbf{r}} = \sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{z}}$$

$$\mathbf{e}_{\perp} = \hat{\mathbf{y}} \quad \mathbf{e}_{\parallel} = -\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{z}}$$

$$\beta = \beta \hat{\mathbf{z}}$$

$$\Delta\beta = \Delta\beta (\cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}})$$

$$\Delta\beta + \hat{\mathbf{r}} \times (\beta \times \Delta\beta) = \Delta\beta (1 - \hat{\mathbf{r}} \cdot \beta) + \beta (\hat{\mathbf{r}} \cdot \Delta\beta)$$

$$\mathbf{e}_{\perp} \cdot (\Delta\beta + \hat{\mathbf{r}} \times (\beta \times \Delta\beta)) = \Delta\beta \sin\phi (1 - \beta \cos\theta)$$

$$\mathbf{e}_{\parallel} \cdot (\Delta\beta + \hat{\mathbf{r}} \times (\beta \times \Delta\beta)) = \Delta\beta \cos\phi (\beta - \cos\theta)$$

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Some details:

$$\beta = \beta \hat{\mathbf{z}} \quad \hat{\mathbf{r}} = \sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{z}}$$

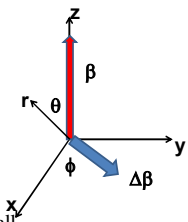
$$\mathbf{e}_{\parallel} = -\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{z}} \quad \mathbf{e}_{\perp} = \hat{\mathbf{y}}$$

$$\Delta\beta = \Delta\beta (\cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}})$$

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Radiation during collisions -- continued
Intensity expressions:

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{(\beta - \cos\theta)^2}{(1 - \beta \cos\theta)^4}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{1}{(1 - \beta \cos\theta)^2}$$


Relativistic collision at low ω and with small $|\Delta\boldsymbol{\beta}|$ and $\Delta\boldsymbol{\beta}$ perpendicular to plane of $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$, as a function of θ where $\hat{\mathbf{r}} \cdot \boldsymbol{\beta} = \beta \cos\theta$;
Integrating over solid angle:

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2$$

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Some details:

$$\int d\Omega \frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 2\pi \int_{-1}^1 d\cos\theta \frac{(\beta - \cos\theta)^2}{(1 - \beta \cos\theta)^4}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{3} \frac{1}{(1 - \beta^2)}$$

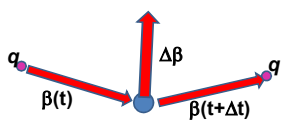
$$\int d\Omega \frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \int_{-1}^1 d\cos\theta \frac{1}{(1 - \beta \cos\theta)^2}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{(1 - \beta^2)}$$

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2$$

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Estimation of $\Delta\boldsymbol{\beta}$



Momentum transfer:

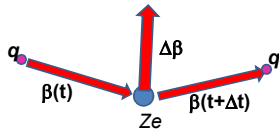
$$Qc \equiv |\mathbf{p}(t + \tau) - \mathbf{p}(t)| c \approx \gamma M c^2 |\Delta\boldsymbol{\beta}|$$

mass of particle having charge q

$$\frac{dI}{d\omega} = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2 \approx \frac{2}{3\pi} \frac{q^2}{M^2 c^3} Q^2$$

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Estimation of $\Delta\beta$ -- for the case of Rutherford scattering

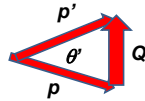


Assume that target nucleus (charge Ze) has mass $\gg M$;
Rutherford scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Ze q}{pv}\right)^2 \frac{1}{(2\sin(\theta'/2))^4}$$

Assuming elastic scattering:

$$Q^2 = (2p\sin(\theta'/2))^2 = 2p^2(1 - \cos\theta')$$

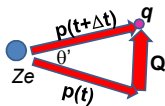


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Case of Rutherford scattering -- continued



Rutherford scattering cross - section :

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Ze q}{pv}\right)^2 \frac{1}{(2\sin(\theta'/2))^4}$$

$$\frac{d\sigma}{dQ} = \int d\phi' \frac{d\sigma}{d\Omega} \frac{d\Omega}{dQ}$$

$$Q^2 = (2p\sin(\theta'/2))^2 = 2p^2(1 - \cos\theta')$$

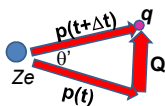
$$\Rightarrow \frac{d\sigma}{dQ} = 8\pi \left(\frac{Ze q}{\beta c}\right)^2 \frac{1}{Q^3}$$

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Case of Rutherford scattering -- continued



Differential radiation cross section :

$$\frac{d^2\chi}{d\omega dQ} = \frac{dI}{d\omega} \frac{d\sigma}{dQ} = \left(\frac{2}{3\pi} \frac{q^2}{M^2 c^3} Q^2\right) \left(8\pi \left(\frac{Ze q}{\beta c}\right)^2 \frac{1}{Q^3}\right)$$

$$= \frac{16}{3} \frac{(Ze)^2}{c} \left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \frac{1}{Q}$$

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Differential radiation cross section -- continued

Integrating over momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16 (Ze)^2}{3 c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{Q_{\max}}{Q_{\min}} \right)$$

Comment on frequency dependence --

Original expression for radiation intensity :

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt e^{i\omega(t - \hat{r} \cdot \mathbf{R}_q(t)/c)} \frac{d}{dt} \left[\frac{\hat{r} \times (\hat{r} \times \boldsymbol{\beta})}{1 - \hat{r} \cdot \boldsymbol{\beta}} \right] \right|^2$$

In the previous derivations, we have assumed that

$$\omega(t - \hat{r} \cdot \mathbf{R}_q(t)/c) \ll 1.$$

$$\omega(t - \hat{r} \cdot \mathbf{R}_q(t)/c) = \omega \left(t - \hat{r} \cdot \int_0^t dt' \boldsymbol{\beta}(t') \right) \approx \omega \tau (1 - \hat{r} \cdot \langle \boldsymbol{\beta} \rangle)$$

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Differential radiation cross section -- continued

Radiation cross section in terms of momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16 (Ze)^2}{3 c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{Q_{\max}}{Q_{\min}} \right)$$

Note that: $Q^2 = 2p^2(1 - \cos\theta') \Rightarrow Q_{\max} = 2p$

In general, Q_{\min} is determined by the collision time

condition $\omega\tau < 1 \Rightarrow Q_{\min} \approx \frac{2Ze q \omega}{v^2}$

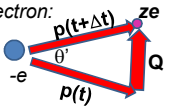
Radiation cross section for classical non - relativistic process

$$\frac{d\chi}{d\omega} = \frac{16 (Ze)^2}{3 c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{\lambda M v^3}{Ze q \omega} \right) \quad \lambda = \text{"fudge factor" of order unity}$$

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Electromagnetic effects in energy loss processes (see Chap. 13 of Jackson)

Again consider Rutherford scattering – now of a nucleus (or alpha particle ze incident on an electron $-e$ in rest frame of electron:



Rutherford scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2pv} \right)^2 \frac{1}{(\sin(\theta'/2))^4}$$

$$\frac{d\sigma}{dQ^2} = \int d\varphi' \frac{d\sigma}{d\Omega} \frac{d\Omega}{dQ^2}$$

$$Q^2 = (2p \sin(\theta'/2))^2 = 2p^2(1 - \cos\theta')$$

$$\Rightarrow \frac{d\sigma}{dQ^2} = 4\pi \left(\frac{ze^2}{\beta c Q^2} \right)^2$$

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Energy loss continued

Let T represent energy loss due to electron of mass m :

$$T = Q^2 / 2m$$

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mc^2 \beta^2 T^2}$$

Estimate of energy loss per unit distance
in the presence of NZ electrons per unit volume

$$\frac{dE}{dx} \approx NZ \int_{\epsilon}^{T_{max}} dT \frac{d\sigma}{dT} \quad \text{minimum energy transfer}$$

$$= 2\pi NZ \frac{z^2 e^4}{mc^2 \beta^2} \ln\left(\frac{2\gamma^2 \beta^2 mc^2}{\epsilon}\right) + (\text{quantum effects})$$

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Energy loss continued

Refining this result, Bethe and Fermi noticed that the analysis lacked consideration of the effects of electromagnetic fields. Representing the colliding electrons in terms of a dielectric function $\epsilon(\omega)$ and the energetic particle of charge ze in terms of the charge and current density:

In Fourier space:

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega) \right] \Phi(\mathbf{k}, \omega) = \frac{4\pi}{\epsilon(\omega)} \rho(\mathbf{k}, \omega)$$

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega) \right] \mathbf{A}(\mathbf{k}, \omega) = \frac{4\pi}{c} \mathbf{J}(\mathbf{k}, \omega)$$

$$\rho(\mathbf{k}, \omega) = \frac{ze}{2\pi} \delta(\omega - \mathbf{v} \cdot \mathbf{k})$$

$$\mathbf{J}(\mathbf{k}, \omega) = \mathbf{v} \rho(\mathbf{k}, \omega)$$

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Energy loss continued

$$\Phi(\mathbf{k}, \omega) = \frac{2ze}{\epsilon(\omega)} \frac{\delta(\omega - \mathbf{v} \cdot \mathbf{k})}{k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)}$$

$$\mathbf{A}(\mathbf{k}, \omega) = \epsilon(\omega) \frac{\mathbf{v}}{c} \Phi(\mathbf{k}, \omega)$$

The energy loss will be calculated
from the work on the electron by the field:

$$\Delta E = -e \int_{-\infty}^{\infty} dt \mathbf{v} \cdot \mathbf{E}(t) = 2e\Re \left(\int_0^{\infty} d\omega i\omega \mathbf{r}(\omega) \cdot \mathbf{E}^*(\omega) \right)$$

The resultant loss estimate is

$$\frac{dE}{dx} \approx \frac{z^2 e^2 \omega_p^2}{2c^2} \ln\left(\frac{2mc^2 \epsilon}{\hbar^2 \omega_p^2}\right) \quad \text{where } \omega_p^2 \equiv \frac{4\pi NZ e^2}{m}$$

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