

**PHY 712 Electrodynamics**  
**9-9:50 AM MWF Olin 105**

**Plan for Lecture 27:**

**Finish Chap. 11 and begin Chap. 14**

**A. Electromagnetic field transformations & corresponding analysis using Liénard-Wiechert potentials for constant velocity sources**

**B. Radiation by moving charged particles**

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23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018				
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018				
30	Wed: 04/04/2018				
31	Fri: 04/06/2018				
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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**Inverse transformation of field strength tensor**

$$F'^{\alpha\beta} = \Lambda^{\alpha\gamma} F^{\gamma\delta} \Lambda_{\delta}^{-1\beta}$$

$$\Lambda_{\nu}^{-1} = \begin{pmatrix} \gamma_{\nu} & -\gamma_{\nu}\beta_{\nu} & 0 & 0 \\ -\gamma_{\nu}\beta_{\nu} & \gamma_{\nu} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_{\nu}(E_y - \beta_{\nu}B_z) & -\gamma_{\nu}(E_z + \beta_{\nu}B_y) \\ E_x & 0 & -\gamma_{\nu}(B_z - \beta_{\nu}E_y) & \gamma_{\nu}(B_y + \beta_{\nu}E_z) \\ \gamma_{\nu}(E_y - \beta_{\nu}B_z) & \gamma_{\nu}(B_z - \beta_{\nu}E_y) & 0 & -B_x \\ \gamma_{\nu}(E_z + \beta_{\nu}B_y) & -\gamma_{\nu}(B_y + \beta_{\nu}E_z) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_{\nu}(E_y - \beta_{\nu}B_z) \qquad B'_y = \gamma_{\nu}(B_y + \beta_{\nu}E_z)$$

$$E'_z = \gamma_{\nu}(E_z + \beta_{\nu}B_y) \qquad B'_z = \gamma_{\nu}(B_z - \beta_{\nu}E_y)$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x \quad B_x = B'_x$$

$$E_y = \gamma_v (E'_y + \beta B'_z) \quad B_y = \gamma_v (B'_y - \beta E'_z)$$

$$E_z = \gamma_v (E'_z - \beta B'_y) \quad B_z = \gamma_v (B'_z + \beta E'_y)$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x = \frac{q(-vt')}{((-vt')^2 + b^2)^{3/2}}$$

$$E_y = \gamma_v (E'_y) = \frac{q(\gamma_v b)}{((-vt')^2 + b^2)^{3/2}}$$

$$B_z = \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-vt')^2 + b^2)^{3/2}}$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x = \frac{q(-v\gamma_v t')}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

$$E_y = \gamma_v (E'_y) = \frac{q(\gamma_v b)}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

$$B_z = \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

Expression in terms of consistent coordinates

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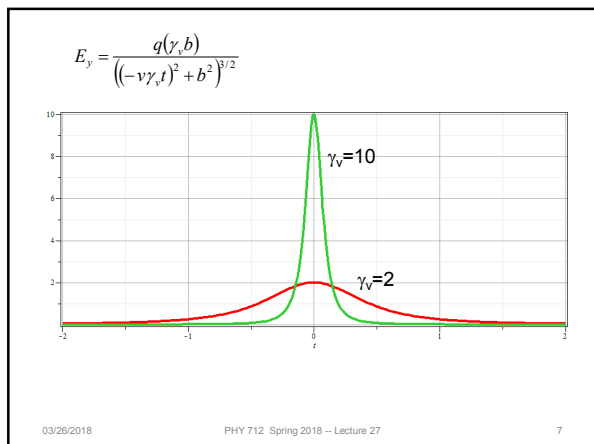
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Examination of this system from the viewpoint of the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) \right]$$

For our example:  
 $\mathbf{R}_r(t_r) = vt_r \hat{\mathbf{x}} \quad \mathbf{r} = b\hat{\mathbf{y}}$   
 $\mathbf{R} = b\hat{\mathbf{y}} - vt_r \hat{\mathbf{x}} \quad R = \sqrt{v^2 t_r^2 + b^2}$   
 $\mathbf{v} = v\hat{\mathbf{x}} \quad t_r = t - \frac{R}{c}$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2}\right) \right]$$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(0, b, 0, t) = q \frac{-v\gamma t \hat{\mathbf{x}} + \gamma b \hat{\mathbf{y}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B}(0, b, 0, t) = q \frac{\gamma b b \hat{\mathbf{z}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

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Some details

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2}\right) \right]$$

For our example:

$$\mathbf{R}_q(t_r) = vt_r \hat{\mathbf{x}} \quad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r \hat{\mathbf{x}} \quad R = \sqrt{v^2 t_r^2 + b^2}$$

$$\mathbf{v} = v\hat{\mathbf{x}} \quad t_r = t - \frac{R}{c}$$

$t_r$  must be a solution to a quadratic equation:

$$t_r - t = -\frac{R}{c} \Rightarrow t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2 / c^2 = 0$$

with the physical solution:

$$t_r = \gamma \left( t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c} \right)$$

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Some details continued:

Now we can express  $R$  as:

$$R = \gamma \left( -\beta v \gamma t + \sqrt{(v\gamma t)^2 + b^2} \right)$$

and the related quantities:

$$\mathbf{R} - \mathbf{v}R / c = -v\gamma t \hat{\mathbf{x}} + b\hat{\mathbf{y}}$$

$$R - \mathbf{v} \cdot \mathbf{R} / c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) \right] = q \frac{-v\gamma t \hat{\mathbf{x}} + \gamma b \hat{\mathbf{y}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2}\right) \right] = q \frac{\gamma \beta b \hat{\mathbf{z}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

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Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

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Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^3} \left[ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left( 1 - \frac{v^2}{c^2} \right) + \left( \mathbf{R} \times \left\{ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^3} \left( 1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$$

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Electric field far from source:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left( R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left\{ \mathbf{R} \times \left[ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

$$\text{Let } \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[ (\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

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Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[ (\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

Note: We have assumed that

$$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r}, t) = 0$$

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Power radiated

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

In the non-relativistic limit:  $\beta \ll 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} |\hat{\mathbf{R}} \times [\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}]|^2 = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

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Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

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Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2$$

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