

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 23:

Continue reading Chap. 9 & 10

A. Electromagnetic waves due to specific sources

B. Dipole radiation examples

C. Scattered radiation

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22	Wed: 03/14/2018	Chap. 9	Harmonic radiation	#14	03/19/2018
23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018				
25	Wed: 03/21/2018				
26	Fri: 03/23/2018				
27	Mon: 03/26/2018				
28	Wed: 03/28/2018				
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018				
30	Wed: 04/04/2018				
31	Fri: 04/06/2018				
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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Review:

Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0; \mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Review:
 Formulation of Maxwell's equations in terms of vector and scalar potentials:
 Lorentz gauge form -- require : $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$
 $-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$
 $-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$
 $\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$
 Note that the Lorentz gauge is consistent with the source continuity condition : $\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0$

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Review:
 Electromagnetic waves from time harmonic sources
 Charge density : $\rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega)e^{-i\omega t})$
 Current density : $\mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega)e^{-i\omega t})$
 \Rightarrow Scalar potential : $\Phi(\mathbf{r}, t) = \Re(\tilde{\Phi}(\mathbf{r}, \omega)e^{-i\omega t})$
 \Rightarrow Vector potential : $\mathbf{A}(\mathbf{r}, t) = \Re(\tilde{\mathbf{A}}(\mathbf{r}, \omega)e^{-i\omega t})$
 For $k \equiv \frac{\omega}{c}$:
 $\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$
 $\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$

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Review:
 Electromagnetic waves from time harmonic sources -- continued:
 Useful expansion :
 $\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$
 $\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$
 $\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$
 $\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$
 $\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$

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Review:

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right) \frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x \ll 1 \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x \gg 1 \Rightarrow h_l(x) \approx (-i)^{l+1} \frac{e^{ix}}{x}$$

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Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case :

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

For fields outside extent of source and $kr' \ll 1$ within the source

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \frac{\mathbf{p}(\omega) e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i}{4\pi\omega\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

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Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\nabla\tilde{\Phi}(\mathbf{r}, \omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr}\right)$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{\text{avg}} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega))$$

$$= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$

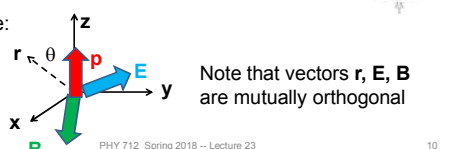
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Properties of dipole radiation field for $kr \gg 1$:

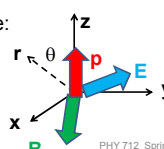
$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}))$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega))$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$


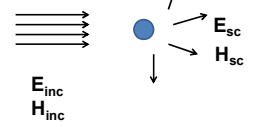
Example:



Note that vectors \mathbf{r} , \mathbf{E} , \mathbf{B} are mutually orthogonal

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Dipole radiation in light scattering by small (dielectric) particles



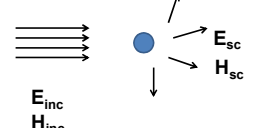
$$\mathbf{E}_{inc} = \hat{\mathbf{e}}_0 E_0 e^{ik_0 \cdot \mathbf{r}} \quad \mathbf{H}_{inc} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{inc}$$

In electric dipole approximation :

$$\mathbf{E}_{sc} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \quad \mathbf{H}_{sc} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{sc}$$

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Dipole radiation in light scattering by small (dielectric) particles



Scattering cross section :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}; \hat{\mathbf{e}}; \hat{\mathbf{k}}_0, \hat{\mathbf{e}}_0) = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{sc} \rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{inc} \rangle_{avg}}$$

$$= \frac{r^2 |\hat{\mathbf{e}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{e}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{e}} \cdot \mathbf{p}|^2$$

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Estimation of scattering dipole moment:
 Suppose the scattering particle is a dielectric sphere with permittivity ϵ and radius a :

$$\mathbf{p} = 4\pi a^3 \epsilon_0 \left(\frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right) \mathbf{E}_{inc} \quad \mathbf{E}_{inc} = \hat{\mathbf{\epsilon}}_0 E_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}}$$

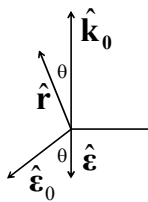
Scattering cross section :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = \frac{r^2 |\hat{\mathbf{\epsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{\epsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{\epsilon}} \cdot \mathbf{p}|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

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Scattering by dielectric sphere with permittivity ϵ and radius a :
 For \mathbf{E}_{inc} polarized in scattering plane:

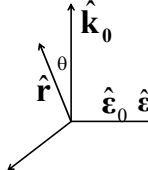


$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 \cos^2 \theta$$

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Scattering by dielectric sphere with permittivity ϵ and radius a :
 For \mathbf{E}_{inc} polarized perpendicular to scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{\epsilon}} \cdot \hat{\mathbf{\epsilon}}_0|^2$$

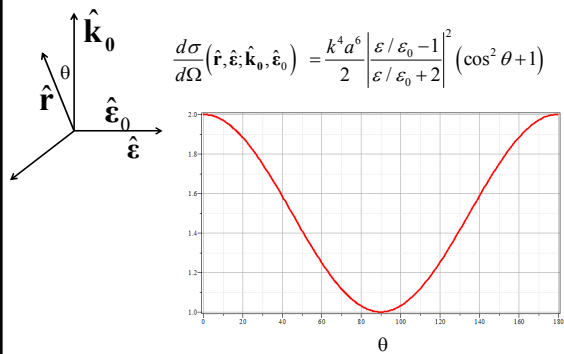
$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2$$

Assuming both polarizations are equally likely, average cross section is given by :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\epsilon}}, \hat{\mathbf{k}}_0, \hat{\mathbf{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

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Scattering by dielectric sphere with permittivity ϵ and radius a :



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Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For $r \gg r'$: $|\mathbf{r}-\mathbf{r}'| \approx r(1 - \hat{\mathbf{r}} \cdot \mathbf{r}'/r + \dots)$

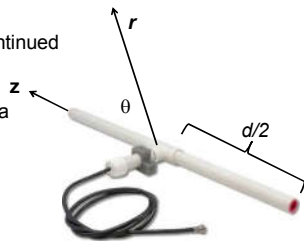
$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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Alternative approach -- continued

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}', \omega) = I_0 \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\delta(y)\hat{\mathbf{z}}$$

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Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi r} \int_{-d/2}^{d/2} dz' e^{-ik \cos(\theta) z'} \sin\left(\frac{kd}{2} - k|z'|\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi kr} \left(\frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2 \theta} \right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

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Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

for $kd = \pi$: $\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$

for $kd = 2\pi$: $\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$

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Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

$kd = \pi$ 2π 3π 4π 5π

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