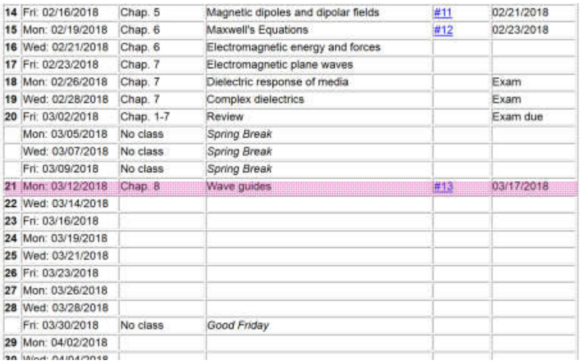
PHY 712 Electrodynamics
9-9:50 AM Olin 105

Plan for Lecture 21:

Chap. 8 in Jackson – Wave Guides

1. TEM, TE, and TM modes
2. Justification for boundary conditions; behavior of waves near conducting surfaces



14	Fri: 02/16/2018	Chap. 5	Magnetic dipoles and dipolar fields	#11	02/21/2018
15	Mon: 02/19/2018	Chap. 6	Maxwell's Equations	#12	02/23/2018
16	Wed: 02/21/2018	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/23/2018	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/26/2018	Chap. 7	Dielectric response of media		Exam
19	Wed: 02/28/2018	Chap. 7	Complex dielectrics		Exam
20	Fri: 03/02/2018	Chap. 1-7	Review		Exam due
	Mon: 03/05/2018	No class	*Spring Break*		
	Wed: 03/07/2018	No class	*Spring Break*		
	Fri: 03/09/2018	No class	*Spring Break*		
21	Mon: 03/12/2018	Chap. 8	Wave guides	#13	03/17/2018
22	Wed: 03/14/2018				
23	Fri: 03/16/2018				
24	Mon: 03/19/2018				
25	Wed: 03/21/2018				
26	Fri: 03/23/2018				
27	Mon: 03/26/2018				
28	Wed: 03/28/2018				
	Fri: 03/30/2018	No class	*Good Friday*		
29	Mon: 04/02/2018				
30	Wed: 04/04/2018				

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E}; \quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law}: \quad \nabla\cdot\mathbf{E}=0$$

$$\text{Ampere-Maxwell's law}: \quad \nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$$

$$\text{Faraday's law}: \quad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$$

$$\text{No magnetic monopoles}: \quad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\left(\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\nabla^2\mathbf{B}-\mu\varepsilon\frac{\partial(\nabla\times\mathbf{E})}{\partial t}$$

$$=-\nabla^2\mathbf{B}+\mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

$$\nabla\times\left(\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\right)=-\nabla^2\mathbf{E}+\frac{\partial(\nabla\times\mathbf{B})}{\partial t}$$

$$=-\nabla^2\mathbf{E}+\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2\mathbf{B}-\frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

$$\nabla^2\mathbf{E}-\frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

$$\text{where} \quad v^2\equiv c^2\frac{\mu_0\varepsilon_0}{\mu\varepsilon}\equiv\frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

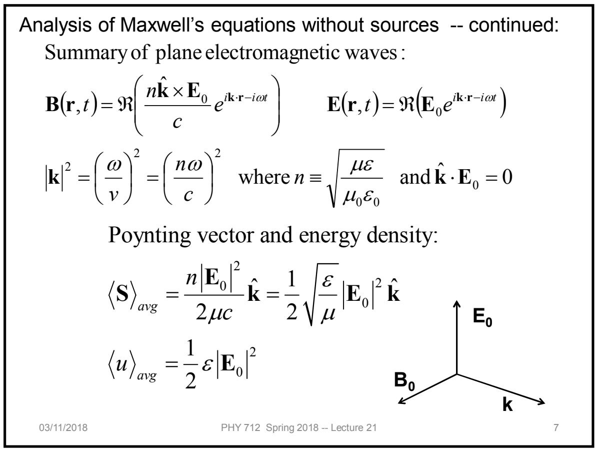
$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

$$\left.\begin{array}{l}\text{Note that } \mathbf{E}_0 \text{ and } \mathbf{B}_0 \text{ are not independent;}\\ \text{from Faraday's law}: \quad \nabla\times\mathbf{E}+\dfrac{\partial\mathbf{B}}{\partial t}=0\\ \qquad\Rightarrow\mathbf{B}_0=\dfrac{\mathbf{k}\times\mathbf{E}_0}{\omega}=\dfrac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}\\ \text{also note}: \quad \hat{\mathbf{k}}\cdot\mathbf{E}_0=0 \quad\text{and}\quad \hat{\mathbf{k}}\cdot\mathbf{B}_0=0\end{array}\right\}$$

For real ε, μ, n, k



Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

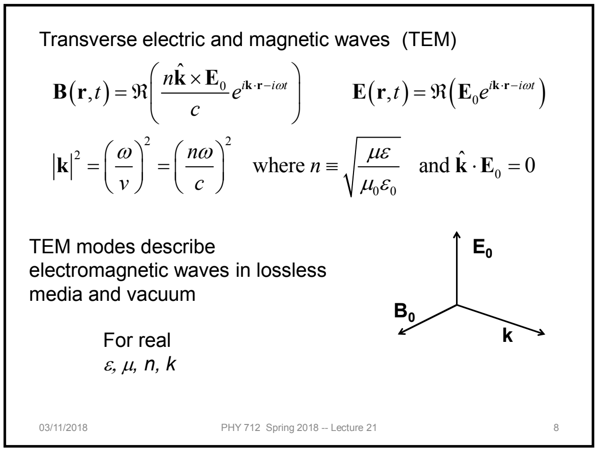
$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle\mathbf{S}\rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

TEM modes describe electromagnetic waves in lossless media and vacuum

For real
ε, μ, n, k

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b\mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{H} = 0$$

$$\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \nabla\times\mathbf{H} = \sigma\mathbf{E} + \varepsilon_b\frac{\partial\mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E},\mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + in_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{in_R(\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t}\right)$$

Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k}=(n_R+in_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=0$$

$$-(n_R+in_I)^2+i\frac{\mu\sigma c^2}{\omega}+\mu\varepsilon_b c^2=0$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R=\omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I=\omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega}>>1 \qquad \frac{\omega}{c}n_R\approx\frac{\omega}{c}n_I\approx\sqrt{\frac{\mu\sigma\omega}{2}}\equiv\frac{1}{\delta}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t)=e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta-i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t)=\frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)=\frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Some representative values of skin depth

Ref: Lorrain[2] and Corson

$$\frac{\omega}{c}n_R\approx\frac{\omega}{c}n_I\approx\sqrt{\frac{\mu\sigma\omega}{2}}\equiv\frac{1}{\delta}$$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\varepsilon_b|\mathbf{E}|^2$

Magnetic energy density: $\mu|\mathbf{H}|^2$

Ratio inside conducting media: $$\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu\left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b\mu\omega^2\delta^2}{2}$$

$$=2\pi^2\frac{\varepsilon_b}{\varepsilon_0}\frac{\mu}{\mu_0}\frac{\delta^2}{\lambda^2}$$

For $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0|\mathbf{E}|^2}{\mu_0|\mathbf{H}|^2} = 1$

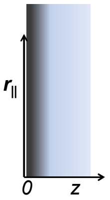
Fields near the surface on an ideal conductor -- continued

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

In this limit, $\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + in_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

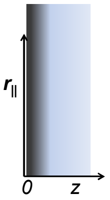
$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Note that the **H** field is larger than **E** field so we can write:

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

Boundary values for ideal conductor

Inside the conductor:

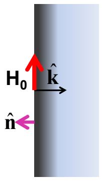
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

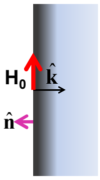
$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$



Wave guides – dielectric media with one or more metal boundary

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$



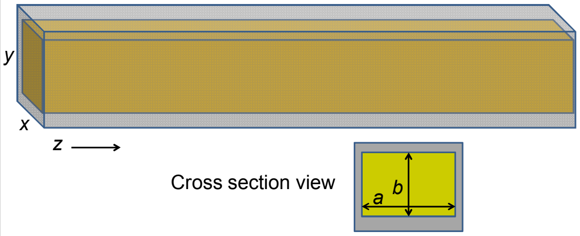
Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

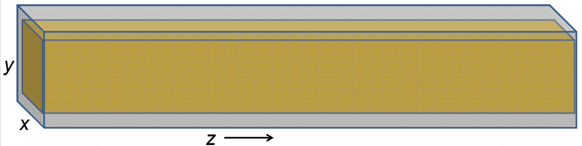
Analysis of rectangular waveguide

Boundary conditions at surface of waveguide:

$\mathbf{E}_{\text{tangential}}=0$, $\mathbf{B}_{\text{normal}}=0$



Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium: (assume ε to be real)

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \nabla\times\mathbf{B} - \varepsilon\frac{\partial\mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\varepsilon\omega^2\right) E_x(x,y) = 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0.$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$$

For TE mode with $E_z \equiv 0$

$$B_x = -\frac{k}{\omega}E_y$$

$$B_y = \frac{k}{\omega}E_x$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \qquad \frac{\partial B_z}{\partial y} - ikB_y = -i\mu\varepsilon\omega E_x.$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y. \qquad ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\varepsilon\omega E_y.$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z. \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\varepsilon\omega E_z.$$

TE modes for retangular wave guide continued:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x = \frac{\omega}{k}B_y = \frac{-i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b}B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y = -\frac{\omega}{k}B_x = \frac{i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a}B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$\mathbf{E}_{\text{tangential}} = 0$ because: $E_x(x,0) = E_x(x,b) = 0$

and $E_y(0,y) = E_y(a,y) = 0.$

$\mathbf{B}_{\text{normal}} = 0$

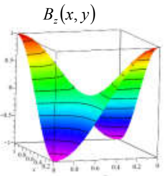
Solution for m=n=1

$$B_z(x,y) = B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$iE_x(x,y) = B_0\left(\frac{\omega n\pi/b}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

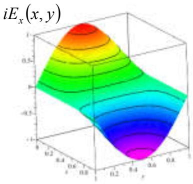
$$iE_y(x,y) = B_0\left(\frac{-\omega m\pi/a}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

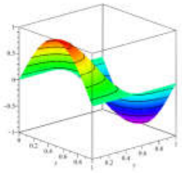
$B_z(x,y)$



$iE_y(x,y)$

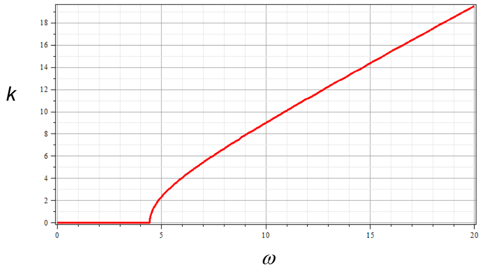
$iE_x(x,y)$

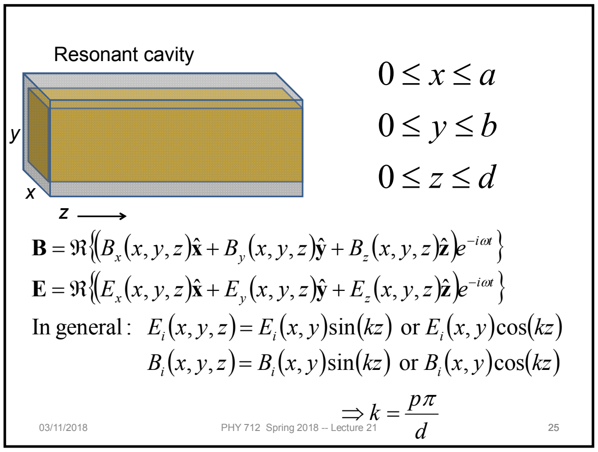




Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$





Resonant cavity

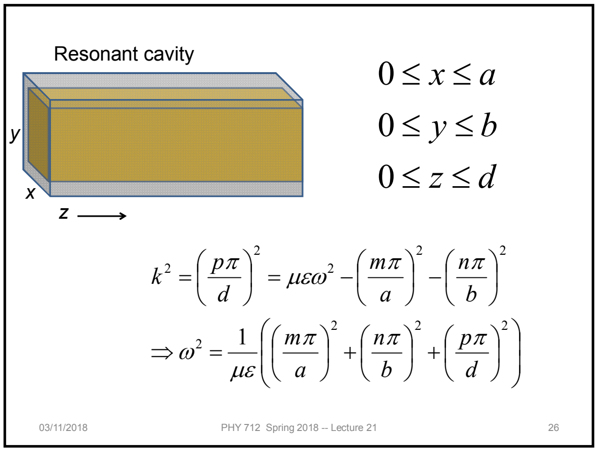
$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$
$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$B_i(x,y,z) = B_i(x,y)\sin(kz)$ or $B_i(x,y)\cos(kz)$

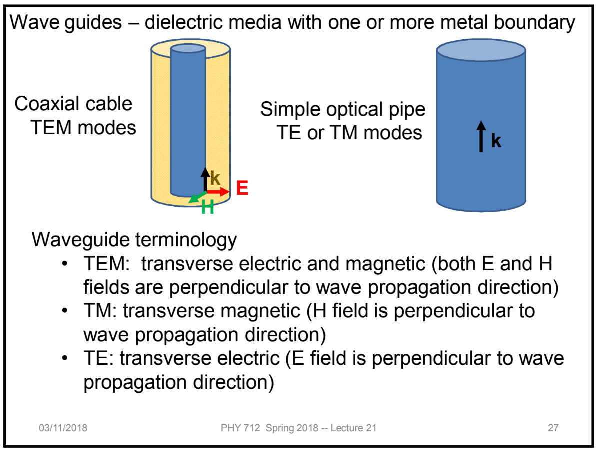
$$\Rightarrow k = \frac{p\pi}{d}$$



Resonant cavity

$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$
$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$



Wave guides – dielectric media with one or more metal boundary

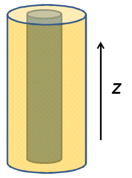
Coaxial cable
TEM modes

Simple optical pipe
TE or TM modes

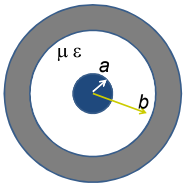
Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Wave guides



Top view:



Inside medium, μ ε assumed to be real

Coaxial cable
TEM modes

(following problem 8.2 in Jackson's text)

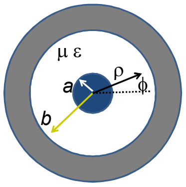
Maxwell's equations inside medium: for $a \le \rho \le b$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = -i\omega\mu\varepsilon \mathbf{E} \qquad \nabla \cdot \mathbf{B} = 0$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Example solution for $a \le \rho \le b$

$$\mathbf{E} = \hat{\boldsymbol{\rho}}\,\Re\left(\frac{E_0 a}{\rho} e^{ikz-i\omega t}\right)$$

$$\mathbf{B} = \hat{\boldsymbol{\varphi}}\,\Re\left(\frac{B_0 a}{\rho} e^{ikz-i\omega t}\right)$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Find:

$$k = \omega\sqrt{\mu\varepsilon}$$

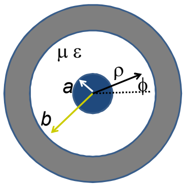
$$E_0 = \frac{B_0}{\sqrt{\mu\varepsilon}}$$

Poynting vector within cable medium (with μ, ε):

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2\mu}\Re\left(\mathbf{E}\times\mathbf{B}^*\right) = \frac{|B_0|^2}{2\mu\sqrt{\mu\varepsilon}}\left(\frac{a}{\rho}\right)^2 \hat{\mathbf{z}}$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Time averaged power in cable material:

$$\int_0^{2\pi} d\phi \int_a^b \rho d\rho \left(\langle \mathbf{S} \rangle_{avg} \cdot \hat{\mathbf{z}}\right) = \frac{|B_0|^2 \pi a^2}{\mu\sqrt{\mu\varepsilon}} \ln\left(\frac{b}{a}\right)$$