PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 5:

Representations, characters, and the "great" orthogonality theorem

Reading: Chapter 3 in DDJ

1. Finish proof of "Great Orthogonality Theorem"
2. Character of a representation
3. Great orthogonality theorem for characters

PHY 745 Group Theory

MWF 11-11:50 AM | OPL 102 | http://www.wfu.edu/~natalie/s17phy745/

Instructor: Natalie Holzwarth | Phone:758-5510 | Office:300 OPL | e-mail:natalie@wfu.edu

Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/20/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 2	Theory of representations		
4	Fri: 01/20/2017	Chap. 2	Proof of the Great Orthonality Theorem	#2	01/23/2017
5	Mon: 01/23/2017	Chap. 3	Notion of character of a representation	#3	01/25/2017
6	Wed: 01/25/2017				
7	Fri: 01/27/2017				

The great orthogonality theorem on unitary irreducible representations

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\mu\nu\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Proof of the great orthogonality theorem

- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

Proof of the great orthogonality theorem

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$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Construct a $\ell_2 \times \ell_1$ matrix

$M \equiv \sum_R \Gamma^2(R) X \Gamma^1(R^{-1})$

Note that: $\Gamma^2(S)M = M\Gamma^1(S)$ where S is a member of the group

For $\ell_2 \neq \ell_1$ Schur's lemma shows that $M = 0$

For $X = 0$ except $X_{\mu\nu} = 1$:

$$\left(\sum_R \Gamma^2(R) X \Gamma^1(R^{-1})\right)_{\alpha\beta} = \sum_R \left(\Gamma^2(R)\right)_{\alpha\mu}\left(\Gamma^1(R^{-1})\right)_{\nu\beta}$$

$$= \sum_R \left(\Gamma^2(R)\right)_{\alpha\mu}\left(\Gamma^1(R)\right)^*_{\beta\nu} = 0$$

Construct a $\ell_2 \times \ell_1$ matrix

$$M \equiv \sum_R \Gamma^2(R) X \Gamma^1(R^{-1})$$

Note that: $\Gamma^2(S)M = M\Gamma^1(S)$ where S is a member of the group

For $2=1$, Schur's lemma shows that $M = sI$ $\quad I \equiv (\ell_1 \times \ell_1)$ identity

$$\left(\sum_R \Gamma^1(R) X \Gamma^1(R^{-1})\right)_{\alpha\beta} = s\delta_{\alpha\beta}$$

For $X = 0$ except for $X_{\mu\nu} = 1$: $\sum_R \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\nu\beta}(R^{-1}) = s_{\mu\nu}\delta_{\alpha\beta}$

Here $s_{\mu\nu}$ denotes the scalar constant for the particular choice of X.

Consider $\alpha = \beta$ and sum over all α:

$$\sum_{R\alpha} \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\nu\alpha}(R^{-1}) = \sum_{R\alpha} \Gamma^1{}_{\nu\alpha}(R^{-1})\Gamma^1{}_{\alpha\mu}(R) = \sum_R \Gamma^1{}_{\nu\mu}(R^{-1}R) = h\Gamma^1{}_{\nu\mu}(E)$$

$$\Rightarrow \sum_{R\alpha} \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\nu\alpha}(R^{-1}) = h\delta_{\mu\nu} = s_{\mu\nu}\sum_\alpha \delta_{\alpha\alpha} = s_{\mu\nu}\ell_1$$

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Proof continued:

$$\sum_R \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\nu\beta}(R^{-1}) = s_{\mu\nu}\delta_{\alpha\beta}$$

$$h\delta_{\mu\nu} = s_{\mu\nu}\ell_1 \qquad \Rightarrow s_{\mu\nu} = \frac{h}{\ell_1}\delta_{\mu\nu}$$

$$\sum_R \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\nu\beta}(R^{-1}) = \sum_R \Gamma^1{}_{\alpha\mu}(R)\left(\Gamma^1(R)\right)^*_{\beta\nu} = \frac{h}{\ell_1}\delta_{\mu\nu}\delta_{\alpha\beta}$$

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

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$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Geometric interpretation:

h dimensional vector space should be spanned by representations $\Gamma^i(R)_{\alpha\beta}$ each of which consists of l_i^2 components.

$$\Rightarrow \sum_i l_i^2 = h$$

For $P(3)$: $\quad 1+1+2^2 = 6$

For $P(4)$: $\quad 1+1+2^2+3^2+3^2 = 24$

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Characters

The character of a representation i for a group element R is defined:

$$\chi^i(R) \equiv \mathrm{Tr}\left(\Gamma^i(R)\right) \equiv \sum_\alpha \left(\Gamma^i(R)\right)_{\alpha\alpha}$$

Note that the character is unique for each irreducible representation.

Suppose $M' = U^\dagger M U$

$\mathrm{Tr}(M') = \mathrm{Tr}(M)$

Details: $\sum_i M'_{ii} = \sum_{ijk} U^\dagger_{ij} M_{jk} U_{ki} = \sum_{ijk} U_{ki} U^\dagger_{ij} M_{jk} = \sum_{jk} \delta_{jk} M_{jk} = \sum_j M_{jj}$

Note that all members of a class has the same character for any given representation.

Details: $\chi^i(X^{-1}RX) = \sum_\alpha \Gamma^i_{\alpha\alpha}(X^{-1}RX)$

$$= \sum_{\alpha\beta\gamma} \Gamma^i_{\alpha\beta}(X^{-1})\Gamma^i_{\beta\gamma}(R)\Gamma^i_{\gamma\alpha}(X) = \sum_{\alpha\beta\gamma} \Gamma^i_{\gamma\alpha}(X)\Gamma^i_{\alpha\beta}(X^{-1})\Gamma^i_{\beta\gamma}(R)$$

$$= \sum_{\beta\gamma} \Gamma^i_{\gamma\beta}(E)\Gamma^i_{\beta\gamma}(R) = \sum_\gamma \Gamma^i_{\gamma\gamma}(R) = \chi^i(R)$$

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Great orthogonality theorem for characters

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Let $\mu = \nu$ and $\alpha = \beta$ and perform summations

$$\sum_{R\mu\alpha} \left(\Gamma^i(R)_{\mu\mu}\right)^* \Gamma^j(R)_{\alpha\alpha} = \frac{h}{l_i}\delta_{ij}\sum_{\mu\alpha}\delta_{\mu\alpha}\delta_{\mu\alpha}$$

$$\sum_R \left(\chi^i(R)\right)^* \chi^j(R) = h\delta_{ij}$$

In terms of classes $\mathcal{C}$, each with $N_{\mathcal{C}}$ elements:

$$\sum_{\mathcal{C}} N_{\mathcal{C}} \left(\chi^i(\mathcal{C})\right)^* \chi^j(\mathcal{C}) = h\delta_{ij}$$

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Example – *P(3)*:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Classes:

$\mathcal{C}_1 = E$

$\mathcal{C}_2 = A, B, C$

$\mathcal{C}_3 = D, F$

$\Gamma^1(A) = \Gamma^1(B) = \Gamma^1(C) = \Gamma^1(D) = \Gamma^1(E) = \Gamma^1(F) = 1$

$\Gamma^2(A) = \Gamma^2(B) = \Gamma^2(C) = -1 \quad \Gamma^2(E) = \Gamma^2(D) = \Gamma^2(F) = 1$

$$\Gamma^3(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Gamma^3(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Gamma^3(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma^3(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma^3(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

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Character table for *P(3):*

$$\Gamma^1(A)=\Gamma^1(B)=\Gamma^1(C)=\Gamma^1(D)=\Gamma^1(E)=\Gamma^1(F)=1$$
$$\Gamma^2(A)=\Gamma^2(B)=\Gamma^2(C)=-1 \quad \Gamma^2(E)=\Gamma^2(D)=\Gamma^2(F)=1$$

$$\Gamma^3(E)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \Gamma^3(A)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \Gamma^3(B)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

$$\Gamma^3(C)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix} \quad \Gamma^3(D)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} \quad \Gamma^3(F)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

Classes: $\mathcal{C}_1 = E \quad \mathcal{C}_2 = A, B, C \quad \mathcal{C}_3 = D, F$

	$\mathcal{C}_1$	$3\mathcal{C}_2$	$2\mathcal{C}_3$
χ^1	1	1	1
χ^2	1	-1	1
χ^3	2	0	-1

Check orthogonality:

$$\sum_{e} N_e \left(\chi^i(\mathcal{C})\right)^* \chi^j(\mathcal{C}) = h\delta_{ij}$$

	$\mathcal{C}_1$	$3\mathcal{C}_2$	$2\mathcal{C}_3$
χ^1	1	1	1
χ^2	1	-1	1
χ^3	2	0	-1

$$\sum_{e} N_e \left(\chi^3(\mathcal{C})\right)^* \chi^2(\mathcal{C}) = 1\cdot 2\cdot 1 + 3\cdot 0\cdot(-1) + 2\cdot(-1)\cdot 1 = 0$$
$$\sum_{e} N_e \left(\chi^3(\mathcal{C})\right)^* \chi^3(\mathcal{C}) = 1\cdot 2\cdot 2 + 3\cdot 0\cdot 0 + 2\cdot(-1)\cdot(-1) = 6$$

Some further conclusions:

$$\sum_{e} N_e \left(\chi^i(\mathcal{C})\right)^* \chi^j(\mathcal{C}) = h\delta_{ij}$$

The characters χ^i behave as a vector space with the dimension equal to the number of classes.

➔The number of characters=the number of classes

Second character identity:

$$\sum_{i} \left(\chi^i(\mathcal{C}_k)\right)^* \chi^i(\mathcal{C}_l) = \frac{h}{N_{\mathcal{C}_k}}\delta_{kl}$$

The regular representation is composed of h x h matrices constructed as follows, shown with the *P(3)* example:

	E	A	B	C	D	F
E^{-1}	E	A	B	C	D	F
A^{-1}	A	E	D	F	B	C
B^{-1}	B	F	E	D	C	A
C^{-1}	C	D	F	E	A	B
D^{-1}	F	B	C	A	E	D
F^{-1}	D	C	A	B	F	E

➔

$$\Gamma^{reg}(E)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\quad \Gamma^{reg}(A)=\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{bmatrix}$$

$$\Gamma^{reg}(B)=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}\quad \Gamma^{reg}(C)=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\end{bmatrix}$$

$$\Gamma^{reg}(D)=\begin{bmatrix}0&0&0&0&1&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix}\quad \Gamma^{reg}(F)=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\end{bmatrix}$$

Regular representation continued –
Note that the regular representation matrices satisfy the multiplication table of the group and have the same class structure as the group.

Since the characters of the irreducible representations form a spanning vector space, we can decompose $\chi^{reg}(R)$ into a linear combination:

$$\chi^{reg}(R)=\sum_i a_i\chi^i(R) \qquad \text{where} \quad a_i=\frac{1}{h}\sum_R\left(\chi^i(R)\right)^*\chi^{reg}(R)$$

By construction: $\chi^{reg}(R)=h\delta_{RE}$

Also note: $\chi^i(E)=\ell_i$ so that $a_i=\ell_i$

$$\Rightarrow \chi^{reg}(R)=\sum_i \ell_i\chi^i(R)$$

$$\chi^{reg}(E)=\sum_i \ell_i\chi^i(E) \qquad \Rightarrow h=\sum_i \ell_i^2$$

Example for *P(3):*

Classes: $\mathcal{C}_1=E \quad \mathcal{C}_2=A,B,C \quad \mathcal{C}_3=D,F$

	$\mathcal{C}_1$	$3\mathcal{C}_2$	$2\mathcal{C}_3$
χ^1	1	1	1
χ^2	1	-1	1
χ^3	2	0	-1

$$\chi^{reg}(R)=\sum_i \ell_i\chi^i(R)$$

$$\chi^{reg}(E)=\sum_i \ell_i\chi^i(E) \qquad \Rightarrow 6=1+1+4$$

$$\chi^{reg}(\mathcal{C}_2)=\sum_i \ell_i\chi^i(\mathcal{C}_2) \qquad \Rightarrow 0=1-1+0$$

$$\chi^{reg}(\mathcal{C}_3)=\sum_i \ell_i\chi^i(\mathcal{C}_3) \qquad \Rightarrow 0=1+1-2$$