PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 4:

The "great" orthogonality theorem

Reading: Chapter 2 in DDJ

1. **Schur's lemmas**
2. **Prove the "Great Orthgonality Theorem"**

PHY 745 Group Theory

MWF 11-11:50 AM | OPL 102 | http://www.wfu.edu/~natalie/s17phy745/

Instructor: Natalie Holzwarth | Phone: 758-5510 | Office: 300 OPL | e-mail: natalie@wfu.edu

Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/20/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 2	Theory of representations		
4	Fri: 01/20/2017	Chap. 2	Proof of the Great Orthonality Theorem	#2	01/23/2017
5	Mon: 01/23/2017				
6	Wed: 01/25/2017				
7	Fri: 01/27/2017				

The great orthogonality theorem on unitary irreducible representations

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\mu\nu\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Proof of the great orthogonality theorem

- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

Proof of the great orthogonality theorem

- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

Schur's lemma part 1:
A matrix *M* which commutes with all of the matrices of an irreducible representation must be a constant matrix: *M=sI* where *s* is a scalar constant and *I* is the identity matrix.

Schur's lemmas part 1:
A matrix *M* which commutes with all of the matrices of an irreducible representation must be a constant matrix: *M=sI* where s is a scalar constant and I is the identity matrix.

Proof:

Suppose M is a diagonal matrix d, the premise becomes

$d\Gamma^i(R) = \Gamma^i(R)d$ for all elements of the group R

$$\Rightarrow \left(d\Gamma^i(R)\right)_{kl} = \left(\Gamma^i(R)d\right)_{kl}$$

$$d_{kk}\left(\Gamma^i(R)\right)_{kl} = \left(\Gamma^i(R)\right)_{kl} d_{ll}$$

$$\left(d_{kk} - d_{ll}\right)\left(\Gamma^i(R)\right)_{kl} = 0 \quad \text{for all } k,l \text{ and for all } R$$

$$\Rightarrow d_{kk} = d_{ll} \qquad \text{if } \Gamma^i(R) \text{ is irreducible}$$

Example irreducible representation:

$$\Gamma^3(E)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \Gamma^3(A)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \Gamma^3(B)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

$$\Gamma^3(C)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix} \quad \Gamma^3(D)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} \quad \Gamma^3(F)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

Extension of proof to more general matrix M :

Note that $M = H_1 + iH_2$

where $H_1 = M + M^\dagger$

$H_2 = -i\left(M - M^\dagger\right)$

In these terms, the premise is:

$(H_1 + iH_2)\Gamma^i(R) = \Gamma^i(R)(H_1 + iH_2)$ for all elements of the group R

$(H_1 + iH_2)\Gamma^i(R) = \Gamma^i(R)(H_1 + iH_2)$ for all elements of the group R

$\Rightarrow H_1\Gamma^i(R) = \Gamma^i(R)H_1$ and $H_2\Gamma^i(R) = \Gamma^i(R)H_2$

Hermitian matrices can be diagonalized by a unitary transformation *U*

$UH_1\Gamma^i(R)U^\dagger = U\Gamma^i(R)H_1U^\dagger$ for all R

$UH_1U^\dagger U\Gamma^i(R)U^\dagger = U\Gamma^i(R)U^\dagger UH_1U^\dagger$

$d\Gamma^{\prime i}(R) = \Gamma^{\prime i}(R)d$ for all R

where $\Gamma^{\prime i}(R)$ is an equivalent representation of the group

The same construction holds for H_2,

Proof of the great orthogonality theorem

- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

Schur's lemma part 2

Consider two irreducible representations of the same group $\Gamma^1(R)$ with dimension ℓ_1 and $\Gamma^2(R)$ with dimensions ℓ_2. Suppose there exists a rectangular $\ell_1 \times \ell_2$ matrix M such that $M\Gamma^1(R) = \Gamma^2(R)M$ for all R. It follows that either $M \equiv 0$ or $\Gamma^1(R)$ and $\Gamma^2(R)$ are equivalent.

Schur's lemma part 2

Consider two irreducible representations of the same group $\Gamma^1(R)$ with dimension ℓ_1 and $\Gamma^2(R)$ with dimensions ℓ_2.

Suppose there exists a rectangular $\ell_1 \times \ell_2$ matrix M such that $M\Gamma^1(R) = \Gamma^2(R)M$ for all R. It follows that either $M \equiv 0$ or $\Gamma^1(R)$ and $\Gamma^2(R)$ are equivalent.

Suppose $M\Gamma^1(R) = \Gamma^2(R)M$ for all R

$$\left(M\Gamma^1(R)\right)^\dagger = \left(\Gamma^2(R)M\right)^\dagger$$

$$\left(\Gamma^1(R)\right)^\dagger M^\dagger = M^\dagger\left(\Gamma^2(R)\right)^\dagger$$

$$\Gamma^1(R^{-1})M^\dagger = M^\dagger\Gamma^2(R^{-1}) \quad \text{for all } R$$

$$\Rightarrow \Gamma^1(R)M^\dagger = M^\dagger\Gamma^2(R) \quad \text{for all } R$$

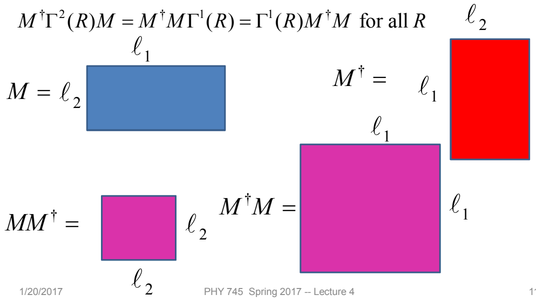
Schur's lemma part 2 -- continued

$M\Gamma^1(R) = \Gamma^2(R)M$ for all R

$\Gamma^1(R)M^\dagger = M^\dagger\Gamma^2(R)$ for all R

$M\Gamma^1(R)M^\dagger = MM^\dagger\Gamma^2(R) = \Gamma^2(R)MM^\dagger$ for all R

$M^\dagger\Gamma^2(R)M = M^\dagger M\Gamma^1(R) = \Gamma^1(R)M^\dagger M$ for all R



Schur's lemma part 2 -- continued

$MM^\dagger\Gamma^2(R) = \Gamma^2(R)MM^\dagger$ for all R

$M^\dagger M\Gamma^1(R) = \Gamma^1(R)M^\dagger M$ for all R

From Schur's first lemma: $MM^\dagger = s_2 I \quad (\ell_2 \times \ell_2)$

$M^\dagger M = s_1 I \quad (\ell_1 \times \ell_1)$

Consider the case where $\ell_1 \neq \ell_2$, it follows that $s_1 = s_2 = 0$

otherwise there is an inconsistency $\left(MM^\dagger\right)^\dagger \neq M^\dagger M$

Consider the case where $\ell_1 = \ell_2$, it follows that $s_1 = s_2{}^*$

For the case that $s_1 \neq 0$:

$M\Gamma^1(R) = \Gamma^2(R)M \quad \Rightarrow \Gamma^1(R) = M^{-1}\Gamma^2(R)M$ for all R

$\Rightarrow$ Representations are equivalent

For the case that $s_1 = 0$:

$MM^\dagger = 0 \quad \Rightarrow M \equiv 0$

Proof of the great orthogonality theorem
- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Construct a $\ell_2 \times \ell_1$ matrix

$$M \equiv \sum_R \Gamma^2(R) X \Gamma^1(R^{-1})$$

Note that: $\Gamma^2(S)M = M\Gamma^1(S)$ where S is a member of the group

$$\begin{aligned}\text{Details: } \Gamma^2(S)M &= \sum_R \Gamma^2(S)\Gamma^2(R)X\Gamma^1(R^{-1})\\ &= \sum_R \Gamma^2(SR)X\Gamma^1(R^{-1})\\ &= \sum_R \Gamma^2(SR)X\Gamma^1(R^{-1})\Gamma^1(S^{-1})\Gamma^1(S)\\ &= \sum_R \Gamma^2(SR)X\Gamma^1((SR)^{-1})\Gamma^1(S)\\ &= M\Gamma^1(S)\end{aligned}$$

Proof continued:

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Construct a $\ell_2 \times \ell_1$ matrix

$$M \equiv \sum_R \Gamma^2(R) X \Gamma^1(R^{-1})$$

Since: $\Gamma^2(S)M = M\Gamma^1(S)$ where S is a member of the group, for $\ell_2 \neq \ell_1$ $M = 0$ for arbitrary X. Choosing particular indices:

$$\left(\sum_R \Gamma^2(R)X\Gamma^1(R^{-1})\right)_{\alpha\beta} = \sum_R \left(\Gamma^2(R)\right)_{\alpha\mu}\left(\Gamma^1(R^{-1})\right)_{\nu\beta}$$

$$= \sum_R \left(\Gamma^2(R)\right)_{\alpha\mu}\left(\Gamma^1(R)\right)^*_{\nu\beta} = 0$$

Proof continued: $\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$

Construct a $\ell_2 \times \ell_1$ matrix

$M \equiv \sum_R \Gamma^2(R) X \Gamma^1(R^{-1})$

Since: $\Gamma^2(S)M = M\Gamma^1(S)$ where S is a member of the group,

for $2=1$ $M = sI$ for arbitrary X. Choosing particular indices:

$\left(\sum_R \Gamma^1(R) X \Gamma^1(R^{-1})\right)_{\alpha\beta} = s\delta_{\alpha\beta}$ *

For particular choice of X : $\sum_{R\mu} \Gamma^1{}_{\alpha\mu}(R)\Gamma^1{}_{\mu\beta}(R^{-1}) = s\delta_{\alpha\beta}\sum_\mu \;\; = s\ell_1\delta_{\alpha\beta}$

$\sum_R \Gamma^1{}_{\alpha\beta}(RR^{-1}) = h\Gamma^1{}_{\alpha\beta}(E) = h\delta_{\alpha\beta} = s\ell_1\delta_{\alpha\beta}$

$\Rightarrow s = \frac{h}{\ell_1}$