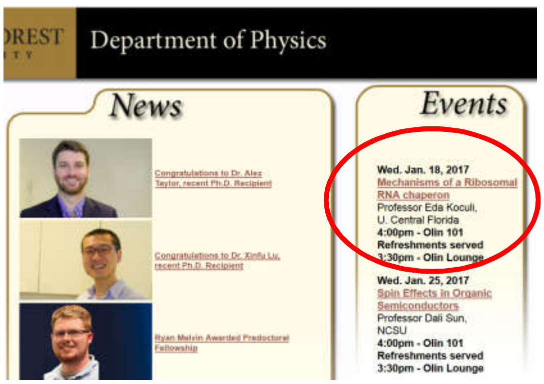
PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 3:

Representation Theory

Reading: Chapter 2 in DDJ

1. Some details of groups and subgroups
2. Preparations for proving the “Great Orthgonality Theorem”



PHY 745 Group Theory

MWF 11-11:50 AM | OPL 102 | http://www.wfu.edu/~natalie/s17phy745/

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Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/20/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 2	Theory of representations		
4	Fri: 01/20/2017				
5	Mon: 01/23/2017				

Groups and subgroups

- A subgroup ***S*** is composed of elements of a group ***G*** which form a group
- A subgroup is called invariant (or normal or self-conjugate) if

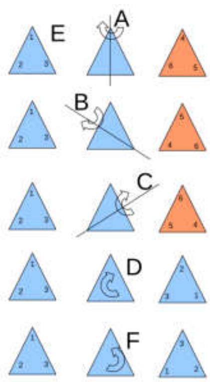
for each element of the original group X, $X^{-1}SX = S$. Your text uses the symbol $\mathcal{N}$ to denote such a normal subgroup.

Example of a 6-member group *E,A,B,C,D,F,G*

Group multiplication table

Group of order 6

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D



Note that this group (called P(3) in your text) can be also described in terms of the permutations:

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Subgroups:

E $\quad (E,A)$ $\quad (E,D,F)$

(E,B)

(E,C)

Invariant subgroup

Classes:

$\mathcal{C}_1 = E$

$\mathcal{C}_2 = A,B,C$

$\mathcal{C}_3 = D,F$

Groups of groups –

The factor group is constructed with respect to a normal subgroup as the collection of its cosets. The factor group is itself a group. Note that for a normal subgroup, the left and right cosets are the same.

Group properties:

Denote group elements by X, Y, E (identity)....

1. Identity of factor group: $E\mathcal{N} = \mathcal{N}$
2. Inverse: $(X\mathcal{N})(X^{-1}\mathcal{N}) = (\mathcal{N}X)(X^{-1}\mathcal{N}) = \mathcal{N}^2 = \mathcal{N}$
3. Multiplication: $(X\mathcal{N})(Y\mathcal{N}) = (XY)\mathcal{N}$
4. Associative property: $\big((X\mathcal{N})(Y\mathcal{N})\big)(Z\mathcal{N}) = (X\mathcal{N})\big((Y\mathcal{N})(Z\mathcal{N})\big)$

P(3) example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Factor group

$$\mathcal{N} = E,D,F$$

$$\mathcal{A} = A,B,C$$

Multiplication table for factor group:

	$\mathcal{N}$	$\mathcal{A}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{A}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{N}$

Representations of a group

A representation of a group is a set of matrices (one for each group element) -- $\Gamma(A), \Gamma(B)...$ that satisfies the multiplication table of the group. The dimension of the matrices is called the dimension of the representation.

Example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Identical Representation:

$$\Gamma^1(A) = \Gamma^1(B) = \Gamma^1(C)$$
$$= \Gamma^1(D) = \Gamma^1(E) = \Gamma^1(F) = 1$$

Another Representation

$$\Gamma^2(A) = \Gamma^2(B) = \Gamma^2(C) = -1$$
$$\Gamma^2(E) = \Gamma^2(D) = \Gamma^2(F) = 1$$

Third Representation

$$\Gamma^3(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Gamma^3(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Gamma^3(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(\mathrm{C}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma^3(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma^3(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

The great orthogonality theorem on unitary irreducible representations

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\mu\nu\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

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Example:

$$\Gamma^1(A)=\Gamma^1(B)=\Gamma^1(C)=\Gamma^1(D)=\Gamma^1(E)=\Gamma^1(F)=1$$

$$\Gamma^2(A)=\Gamma^2(B)=\Gamma^2(C)=-1 \quad \Gamma^2(E)=\Gamma^2(D)=\Gamma^2(F)=1$$

$$\Gamma^3(E)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \Gamma^3(A)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \Gamma^3(B)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

$$\Gamma^3(C)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix} \quad \Gamma^3(D)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} \quad \Gamma^3(F)=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

$$\sum_R \Gamma^1(R)\Gamma^1(R)=6 \qquad \sum_R \Gamma^2(R)\Gamma^2(R)=6 \qquad \sum_R \Gamma^1(R)\Gamma^2(R)=0$$

Special types of matrices -- unitary

$$UU^\dagger = U^\dagger U = 1$$

Example: $U=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

$$UU^\dagger=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Example: $U=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

$$UU^\dagger=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Special types of matrices -- Hermitian

$$H^\dagger = H$$

Example: $H=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=H^\dagger$

For every Hermitian matrix, there is a unitary matrix U which can be used to transformation it into diagonal form:

$$d = U^\dagger H U$$

For our example: $U=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \quad U^\dagger=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$

$$U^\dagger H U=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Here the diagonal elements are real.

Digression: Note that unitary matrices themselves can also be put into diagonal form with a unitary similarity transformation. Last week we had the example:

$$\text{Let} \quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \quad U^\dagger = U^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$U^\dagger M U = U^\dagger \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\theta} & \frac{1}{\sqrt{2}} e^{i\theta} \\ \frac{i}{\sqrt{2}} e^{-i\theta} & \frac{-i}{\sqrt{2}} e^{i\theta} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

In this case the diagonal elements have modulus unity.

Proof of the great orthogonality theorem

- Prove that all representations can be unitary matrices
- Prove Schur's lemma part 1 – any matrix which commutes with all matrices of an irreducible representation must be a constant matrix
- Prove Schur's lemma part 2
- Put all parts together

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Note that for any representation $\Gamma(R)$ of group elements R, the similarity transformed representation $\Gamma'(R) = S^{-1}\Gamma(R)S$ is also a representation.

Prove that all representations can be unitary matrices

Note that for any representation $\Gamma(R)$ of group elements R, the similarity transformed representation $\Gamma'(R) = S^{-1}\Gamma(R)S$ is also a representation.

Suppose that representation $\Gamma(R)$ is not unitary: $\Gamma(R)(\Gamma(R))^{\dagger} \neq 1$

Find S such that $\Gamma'(R) = S^{-1}\Gamma(R)S$ and $\Gamma'(R)(\Gamma'(R))^{\dagger} = 1$.

Answer: $S = Ud^{1/2}$

where $d = U^{\dagger}\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)U$

Answer: $S = Ud^{1/2}$

where $d = U^{\dagger}\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)U$

Summation over all elements of group

Note that $\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)^{\dagger} = \left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)$

Details: $\left(AA^{\dagger}\right)_{ij} = \sum_k A_{ik}A^{\dagger}_{kj} = \sum_k A_{ik}A^{*}_{jk}$

$$\left(AA^{\dagger}\right)^{\dagger}_{ij} = \left(AA^{\dagger}\right)^{*}_{ji} = \sum_k \left(A_{jk}A^{*}_{ik}\right)^{*} = \sum_k A^{*}_{jk}A_{ik}$$

Since $\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)$ is Hermitian, we can find a unitary matrix U to form $d = U^{\dagger}\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)U$

Note that all elements of the diagonal matrix *d* are real and positive.

Choose: $S = Ud^{1/2}$ and construct $\Gamma'(R) = S^{-1}\Gamma(R)S$

$$\Gamma'(R)(\Gamma'(R))^{\dagger} = S^{-1}\Gamma(R)S\left(S^{-1}\Gamma(R)S\right)^{\dagger}$$

$$S^{-1} = d^{-1/2}U^{\dagger} \qquad S^{\dagger} = d^{1/2}U^{\dagger}$$

$$S^{-1}\Gamma(R)S\left(S^{-1}\Gamma(R)S\right)^{\dagger} = d^{-1/2}U^{\dagger}\Gamma(R)Ud^{1/2}d^{1/2}U^{\dagger}\Gamma^{\dagger}(R)Ud^{-1/2}$$

$$= d^{-1/2}U^{\dagger}\Gamma(R)UdU^{\dagger}\Gamma^{\dagger}(R)Ud^{-1/2}$$

$$\Gamma'(R) = S^{-1}\Gamma(R)S$$

$$\Gamma'(R)(\Gamma'(R))^{\dagger} = S^{-1}\Gamma(R)S\left(S^{-1}\Gamma(R)S\right)^{\dagger}$$

$$S^{-1}\Gamma(R)S\left(S^{-1}\Gamma(R)S\right)^{\dagger} = d^{-1/2}U^{\dagger}\Gamma(R)Ud^{1/2}d^{1/2}U^{\dagger}\Gamma^{\dagger}(R)Ud^{-1/2}$$
$$= d^{-1/2}U^{\dagger}\Gamma(R)UdU^{\dagger}\Gamma^{\dagger}(R)Ud^{-1/2}$$

$$d = U^{\dagger}\left(\sum_R \Gamma(R)\Gamma^{\dagger}(R)\right)U$$

$$S^{-1}\Gamma(R)S\left(S^{-1}\Gamma(R)S\right) = d^{-1/2}U^{\dagger}\Gamma(R)\left(\sum_{R'}\Gamma(R')\Gamma^{\dagger}(R')\right)\Gamma^{\dagger}(R)Ud^{-1/2}$$
$$= d^{-1/2}U^{\dagger}\left(\sum_{R''}\Gamma(R'')\Gamma^{\dagger}(R'')\right)Ud^{-1/2}$$
$$= d^{-1/2}dd^{-1/2} = 1$$

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Proved that all representations can be unitary matrices

Consider our example:

$$\Gamma^3(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Gamma^3(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Gamma^3(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma^3(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma^3(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Consider the construction: $\sum_R \Gamma(R)\Gamma^{\dagger}(R)$:

For this example, $\sum_R \Gamma(R)\Gamma^{\dagger}(R) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

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