PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 36:

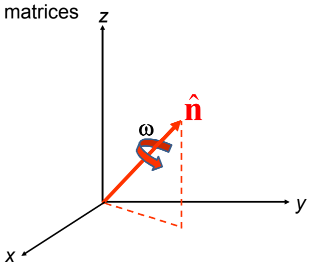
1. **Summary of what we learned about linear Lie groups, especially SO(3) and SU(2), their direct products and Clebsch-Gordan coefficients**
2. **Review of topics in Group Theory**
3. **Course evaluation**

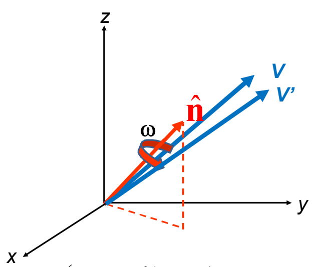
Ref. J. F. Cornwell, *Group Theory in Physics*, Vol I and II, Academic Press (1984)

23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017		Introduction to Lie groups		
33	Wed: 04/12/2017		Introduction to Lie groups		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017		Introduction to Lie groups		
35	Wed: 04/19/2017		Introduction to Lie groups		
36	Fri: 04/21/2017		Introduction to Lie groups		
	Mon: 04/24/2017		Presentations I Jason and Ahmad		
	Wed: 04/26/2017		Presentations II Taylor		

SO(3) – all continuous linear transformations in 3-dimensional space which preserve the length of coordinate vectors -- $R(\omega,\hat{\mathbf{n}})$

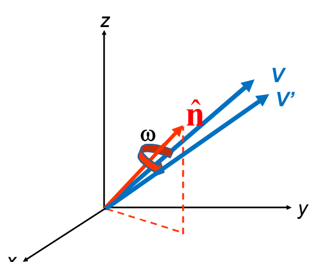
SU(2) – all 2x2 unitary matrices $\mathbf{u}(\omega,\hat{\mathbf{n}})$





$$R(\omega,\hat{\mathbf{n}})=\begin{pmatrix} \cos\omega+n_1^2(1-\cos\omega) & n_3\sin\omega+n_1n_2(1-\cos\omega) & -n_2\sin\omega+n_1n_3(1-\cos\omega) \\ -n_3\sin\omega+n_1n_2(1-\cos\omega) & \cos\omega+n_2^2(1-\cos\omega) & n_1\sin\omega+n_2n_3(1-\cos\omega) \\ n_2\sin\omega+n_1n_3(1-\cos\omega) & -n_1\sin\omega+n_2n_3(1-\cos\omega) & \cos\omega+n_3^2(1-\cos\omega) \end{pmatrix}$$

For example: $R(\omega,\hat{\mathbf{z}})=\begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad -\pi<\omega\le\pi$



$$\mathbf{u}(\omega,\hat{\mathbf{n}})=\begin{pmatrix} \cos\frac{\omega}{2}+in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2}-in_3\sin\frac{\omega}{2} \end{pmatrix}$$

For example: $\mathbf{u}(\omega,\hat{\mathbf{z}})=\begin{pmatrix} e^{i\omega/2} & 0 \\ 0 & e^{-i\omega/2} \end{pmatrix}$ $\quad -2\pi<\omega\le 2\pi$

Class structure of these groups

$$R(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\omega + n_1^2(1-\cos\omega) & n_3\sin\omega + n_1n_2(1-\cos\omega) & -n_2\sin\omega + n_1n_3(1-\cos\omega) \\ -n_3\sin\omega + n_1n_2(1-\cos\omega) & \cos\omega + n_2^2(1-\cos\omega) & n_1\sin\omega + n_2n_3(1-\cos\omega) \\ n_2\sin\omega + n_1n_3(1-\cos\omega) & -n_1\sin\omega + n_2n_3(1-\cos\omega) & \cos\omega + n_3^2(1-\cos\omega) \end{pmatrix}$$

$$\mathrm{Tr}\left(R(\omega,\hat{\mathbf{n}})\right) = 3\cos\omega + \left(n_1^2+n_2^2+n_3^2\right)(1-\cos\omega) = 1 + 2\cos\omega \;\Rightarrow \text{depends on } |\omega| \text{ and not on } \hat{\mathbf{n}}$$

$$\mathbf{u}(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\frac{\omega}{2} + in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} - in_3\sin\frac{\omega}{2} \end{pmatrix}$$

$$\mathrm{Tr}\left(\mathbf{u}(\omega,\hat{\mathbf{n}})\right) = 2\cos\frac{\omega}{2} \quad \text{depends on } |\omega| \text{ and not on } \hat{\mathbf{n}}$$

"Generators" of the groups

$$R(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\omega + n_1^2(1-\cos\omega) & n_3\sin\omega + n_1n_2(1-\cos\omega) & -n_2\sin\omega + n_1n_3(1-\cos\omega) \\ -n_3\sin\omega + n_1n_2(1-\cos\omega) & \cos\omega + n_2^2(1-\cos\omega) & n_1\sin\omega + n_2n_3(1-\cos\omega) \\ n_2\sin\omega + n_1n_3(1-\cos\omega) & -n_1\sin\omega + n_2n_3(1-\cos\omega) & \cos\omega + n_3^2(1-\cos\omega) \end{pmatrix}$$

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{R(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}$$

$$\mathbf{u}(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\frac{\omega}{2} + in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} - in_3\sin\frac{\omega}{2} \end{pmatrix}$$

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{\mathbf{u}(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} in_3 & in_1+n_2 \\ in_1-n_2 & -in_3 \end{pmatrix}$$

Irreducible representations of SO(3) and SU(2) -- focusing on SU(2)

Consider the generator matrices

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{\mathbf{u}(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} in_3 & in_1+n_2 \\ in_1-n_2 & -in_3 \end{pmatrix}$$

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$

In order to determine irreducible representations of SU(2), determine eigenstates associated with generators.

Eigenfunctions associated with generator functions of SU(2)

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$

It is convenient to map these generators into Hermitian matrices

$$\mathbf{A}_p \equiv -i\mathbf{a}_p = \tfrac{1}{2}\boldsymbol{\sigma}_p$$

$$\mathbf{A}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathbf{A}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Commutation relations: $[\mathbf{A}_1, \mathbf{A}_2] = i\mathbf{A}_3$

Define $\mathbf{A}^2 = \mathbf{A}_1^2 + \mathbf{A}_2^2 + \mathbf{A}_3^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{A}_- = \mathbf{A}_1 - i\mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_+ = \mathbf{A}_1 + i\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Eigenfunctions associated with generator functions of SU(2)

Relationships between operators:

$$[\mathbf{A}^2, \mathbf{A}_p] = 0$$

$$\mathbf{A}_-\mathbf{A}_+ = \mathbf{A}^2 - \mathbf{A}_3^2 - \mathbf{A}_3$$

$$\mathbf{A}_+\mathbf{A}_- = \mathbf{A}^2 - \mathbf{A}_3^2 + \mathbf{A}_3$$

Note that this "algebra" is identical to that of the total angular momentum operators

$$\mathbf{J}_x = \hbar\mathbf{A}_1 \quad \mathbf{J}_y = \hbar\mathbf{A}_2 \quad \mathbf{J}_z = \hbar\mathbf{A}_3$$

$\Rightarrow$ Leap to eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$: $|jm\rangle$

$$\mathbf{A}^2 |jm\rangle = j(j+1)|jm\rangle$$

$$\mathbf{A}_3 |jm\rangle = m|jm\rangle$$

Eigenfunctions associated with generator functions of SU(2)

Additional relationships:

$$\mathbf{A}_- |jm\rangle = \{(j+m)(j-m+1)\}^{1/2}|j(m-1)\rangle$$

$$\mathbf{A}_+ |jm\rangle = \{(j+m)(j+m+1)\}^{1/2}|j(m+1)\rangle$$

Mapping the representations back to the original generators of SU(2): $\mathbf{a}_p = i\mathbf{A}_p$

$$D^j_{mm'}(\mathbf{a}_p) \equiv \langle jm|\mathbf{a}_p|jm'\rangle$$

$j = 0$ $\quad D^j_{mm'}(\mathbf{a}_p) = 0$

$j = \tfrac{1}{2}$ $\quad D^j_{mm'}(\mathbf{a}_1) = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad D^j_{mm'}(\mathbf{a}_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad D^j_{mm'}(\mathbf{a}_3) = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Comment of analogous equations for SO(3) --

Eigenfunctions associated with generator functions of SO(3)

$$R(\omega,\hat{\mathbf{n}})=\begin{pmatrix}\cos\omega+n_1^2(1-\cos\omega) & n_3\sin\omega+n_1n_2(1-\cos\omega) & -n_2\sin\omega+n_1n_3(1-\cos\omega)\\ -n_3\sin\omega+n_1n_2(1-\cos\omega) & \cos\omega+n_2^2(1-\cos\omega) & n_1\sin\omega+n_2n_3(1-\cos\omega)\\ n_2\sin\omega+n_1n_3(1-\cos\omega) & -n_1\sin\omega+n_2n_3(1-\cos\omega) & \cos\omega+n_3^2(1-\cos\omega)\end{pmatrix}$$

$$\mathbf{a}=\lim_{\omega\to 0}\left(\frac{R(\omega,\hat{\mathbf{n}})-1}{\omega}\right)=n_1\mathbf{a}_1+n_2\mathbf{a}_2+n_3\mathbf{a}_3=\begin{pmatrix}0 & n_3 & -n_2\\ -n_3 & 0 & n_1\\ n_2 & -n_1 & 0\end{pmatrix}$$

$$\mathbf{a}_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\qquad \mathbf{a}_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\qquad \mathbf{a}_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

$$[\mathbf{a}_1,\mathbf{a}_2]=-\mathbf{a}_3$$

$$\mathbf{a}_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\qquad \mathbf{a}_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\qquad \mathbf{a}_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1,\mathbf{a}_2]=-\mathbf{a}_3$

It is convenient to map these generators into Hermitian matrices

$\mathbf{A}_p\equiv -i\mathbf{a}_p$

$$\mathbf{A}_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\qquad \mathbf{A}_2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}\qquad \mathbf{A}_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

Commutation relations: $[\mathbf{A}_1,\mathbf{A}_2]=i\mathbf{A}_3$

Define $\mathbf{A}^2=\mathbf{A}_1^2+\mathbf{A}_2^2+\mathbf{A}_3^2=2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

However, in this case $\mathbf{A}_3$ is not diagonal; using a similarity transformation:

$$\tilde{\mathbf{A}}_3\equiv U^\dagger\mathbf{A}_3U=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-i}{\sqrt2}&0\\0&0&1\\\frac{-1}{\sqrt2}&\frac{-i}{\sqrt2}&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{i}{\sqrt2}&0&\frac{i}{\sqrt2}\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$\tilde{\mathbf{A}}_1\equiv U^\dagger\mathbf{A}_1U=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-i}{\sqrt2}&0\\0&0&1\\\frac{-1}{\sqrt2}&\frac{-i}{\sqrt2}&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{i}{\sqrt2}&0&\frac{i}{\sqrt2}\\0&1&0\end{pmatrix}=\begin{pmatrix}0&\frac{-i}{\sqrt2}&0\\\frac{i}{\sqrt2}&0&\frac{-i}{\sqrt2}\\0&\frac{i}{\sqrt2}&0\end{pmatrix}$$

$$\tilde{\mathbf{A}}_2\equiv U^\dagger\mathbf{A}_2U=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-i}{\sqrt2}&0\\0&0&1\\\frac{-1}{\sqrt2}&\frac{-i}{\sqrt2}&0\end{pmatrix}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{i}{\sqrt2}&0&\frac{i}{\sqrt2}\\0&1&0\end{pmatrix}=\begin{pmatrix}0&\frac{1}{\sqrt2}&0\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&0\end{pmatrix}$$

In accordance with the usual convention:

$\mathbf{J}_z=\hbar\tilde{\mathbf{A}}_3\qquad \mathbf{J}_y=\hbar\tilde{\mathbf{A}}_1\qquad \mathbf{J}_x=\hbar\tilde{\mathbf{A}}_2$

In general, we can form a representation for SU(2) by evaluating $e^{\omega \mathbf{D}^j(\mathbf{a}_3)}$:

The character of this representation is

$$\chi^j(\omega) = \sum_{m=-j}^{j} e^{im\omega} = \frac{\sin\left(\left(j+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Direct product of irreducible representations $\mathbf{D}^{j_1}$ and $\mathbf{D}^{j_2}$

It can be shown that $\mathbf{D}^{j_1 \otimes j_2} = \mathbf{D}^{j_1+j_2} \oplus \mathbf{D}^{j_1+j_2-1} \ldots \mathbf{D}^{|j_1-j_2|}$

This follows from the character relationships:

$$\chi^{j_1 \otimes j_2}(\omega) = \frac{\sin\left(\left(j_1+\frac{1}{2}\right)\omega\right)\sin\left(\left(j_2+\frac{1}{2}\right)\omega\right)}{\sin^2\left(\frac{1}{2}\omega\right)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega)$$

For example: $j_1=1/2$ and $j_2=1/2$

It can be shown that $\mathbf{D}^{j_1 \otimes j_2} = \mathbf{D}^{j_1+j_2} \oplus \mathbf{D}^{j_1+j_2-1} \ldots \mathbf{D}^{|j_1-j_2|}$

This follows from the character relationships:

$$\chi^{1/2 \otimes 1/2}(\omega) = \chi^1(\omega) \oplus \chi^0(\omega)$$

$$\left\langle j_1 m_1 j_2 m_2 \middle| j_1 j_2 j m \right\rangle = \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 1 \right\rangle = 1$$

$$\left\langle \tfrac{1}{2} -\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 0 \right\rangle = \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} -\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 0 \right\rangle = \tfrac{1}{\sqrt{2}}$$

$$\left\langle \tfrac{1}{2} -\tfrac{1}{2}\tfrac{1}{2} -\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 -1 \right\rangle = 1$$

$$\left\langle \tfrac{1}{2} -\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 0 0 \right\rangle = -\left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} -\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 0 0 \right\rangle = \tfrac{1}{\sqrt{2}}$$

Review --

Group theory

An abstract algebraic construction in mathematics

Definition of a group:

A group is a collection of "elements" – $A, B, C, \ldots$ and a "multiplication" process. The abstract multiplication ($\cdot$) pairs two group elements, and associates the "result" with a third element. (For example ($A \cdot B = C$).) The elements and the multiplication process must have the following properties.

1. The collection of elements is closed under multiplication. That is, if elements A and B are in the group and $A \cdot B = C$, element C must be in the group.
2. One of the members of the group is a "unit element" (E). That is, for any element A of the group, $A \cdot E = E \cdot A = A$.
3. For each element A of the group, there is another element A^{-1} which is its "inverse". That is $A \cdot A^{-1} = A^{-1} \cdot A = E$.
4. The multiplication process is "associative". That is for sequential mulplication of group elements A, B, and C, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Definition:

An element $B \equiv XAX^{-1}$ is defined as conjugate to element A, where X is any element of the group.

Definition:

A class is composed of members of a group which are generated by the conjugate construction:

$\mathcal{C} = X_i^{-1} Y X_i$ where Y is a fixed group element and X_i are all of the elements of the group.

The great orthogonality theorem

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\mu\nu\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Character of a representation:

$$\chi^j(R) = \sum_\mu \Gamma^j(R)_{\mu\mu}$$

In terms of classes $\mathcal{C}$, each with $N_{\mathcal{C}}$ elements:

$$\sum_{\mathcal{C}} N_{\mathcal{C}} \left(\chi^i(\mathcal{C})\right)^* \chi^j(\mathcal{C}) = h\delta_{ij}$$

Some further conclusions:

$$\sum_{e} N_e \left(\chi^i(\mathcal{C})\right)^* \chi^j(\mathcal{C}) = h\delta_{ij}$$

The characters χ^i behave as a vector space with the dimension equal to the number of classes.

→The number of characters=the number of classes

Second character identity:

$$\sum_{i} \left(\chi^i(\mathcal{C}_k)\right)^* \chi^i(\mathcal{C}_l) = \frac{h}{N_{\mathcal{C}_k}}\delta_{kl}$$

Example of character table for D_3

$D_3(32)$			E	$2C_3$	$3C_2'$
x^2+y^2, z^2		A_1	1	1	1
	R_z, z	A_2	1	1	-1
(xz, yz), (x^2-y^2, xy)	(x, y), (R_x, R_y)	E	2	-1	0

"Standard" notation for representations of D_3

Extension of group notions to continuous groups

- Introduced notion of mapping group members to finite number of continuous parameters
- Introduced notion of "metric"
- Introduced notion of connected subgroups
- "Algebraic" properties of group generators