PHY 745 Group Theory
11-11:50 AM MWF Olin 102

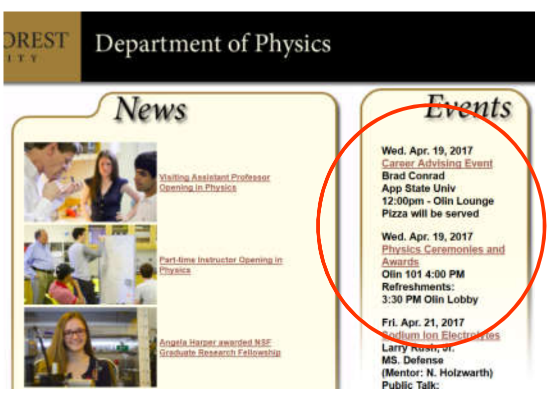
Plan for Lecture 35:

Introduction to linear Lie groups
Examples – SO(3) and SU(2)

1. General properties
2. Class structures
3. Representations
4. Direct products; Clebsch-Gordan coefficients
5. Examples

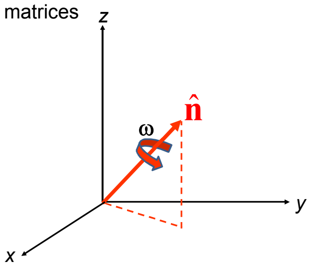
Ref. J. F. Cornwell, *Group Theory in Physics*, Vol I and II, Academic Press (1984)

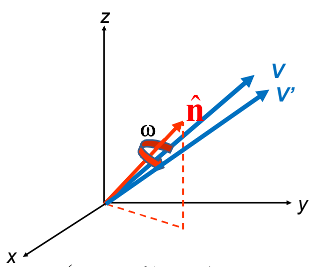
23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017		Introduction to Lie groups		
33	Wed: 04/12/2017		Introduction to Lie groups		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017		Introduction to Lie groups		
35	Wed: 04/19/2017		Introduction to Lie groups		
36	Fri: 04/21/2017		Introduction to Lie groups		
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



SO(3) – all continuous linear transformations in 3-dimensional space which preserve the length of coordinate vectors -- $R(\omega,\hat{\mathbf{n}})$

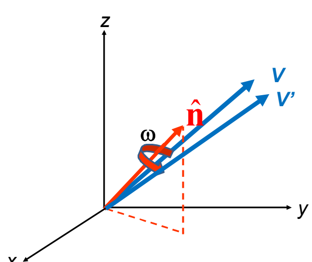
SU(2) – all 2x2 unitary matrices $\mathbf{u}(\omega,\hat{\mathbf{n}})$





$$R(\omega,\hat{\mathbf{n}})=\begin{pmatrix} \cos\omega+n_1^2(1-\cos\omega) & n_3\sin\omega+n_1n_2(1-\cos\omega) & -n_2\sin\omega+n_1n_3(1-\cos\omega) \\ -n_3\sin\omega+n_1n_2(1-\cos\omega) & \cos\omega+n_2^2(1-\cos\omega) & n_1\sin\omega+n_2n_3(1-\cos\omega) \\ n_2\sin\omega+n_1n_3(1-\cos\omega) & -n_1\sin\omega+n_2n_3(1-\cos\omega) & \cos\omega+n_3^2(1-\cos\omega) \end{pmatrix}$$

For example: $R(\omega,\hat{\mathbf{z}})=\begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad -\pi<\omega\le\pi$



$$\mathbf{u}(\omega,\hat{\mathbf{n}})=\begin{pmatrix} \cos\frac{\omega}{2}+in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2}-in_3\sin\frac{\omega}{2} \end{pmatrix}$$

For example: $\mathbf{u}(\omega,\hat{\mathbf{z}})=\begin{pmatrix} e^{i\omega/2} & 0 \\ 0 & e^{-i\omega/2} \end{pmatrix}$ $\quad -2\pi<\omega\le 2\pi$

Class structure of these groups

$$R(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\omega + n_1^2(1-\cos\omega) & n_3\sin\omega + n_1n_2(1-\cos\omega) & -n_2\sin\omega + n_1n_3(1-\cos\omega) \\ -n_3\sin\omega + n_1n_2(1-\cos\omega) & \cos\omega + n_2^2(1-\cos\omega) & n_1\sin\omega + n_2n_3(1-\cos\omega) \\ n_2\sin\omega + n_1n_3(1-\cos\omega) & -n_1\sin\omega + n_2n_3(1-\cos\omega) & \cos\omega + n_3^2(1-\cos\omega) \end{pmatrix}$$

$$\mathrm{Tr}\left(R(\omega,\hat{\mathbf{n}})\right) = 3\cos\omega + \left(n_1^2+n_2^2+n_3^2\right)(1-\cos\omega) = 1+2\cos\omega \quad \Rightarrow \text{depends on } |\omega| \text{ and not on } \hat{\mathbf{n}}$$

$$\mathbf{u}(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\frac{\omega}{2} + in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} - in_3\sin\frac{\omega}{2} \end{pmatrix}$$

$$\mathrm{Tr}\left(\mathbf{u}(\omega,\hat{\mathbf{n}})\right) = 2\cos\frac{\omega}{2} \quad \text{depends on } |\omega| \text{ and not on } \hat{\mathbf{n}}$$

"Generators" of the groups

$$R(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\omega + n_1^2(1-\cos\omega) & n_3\sin\omega + n_1n_2(1-\cos\omega) & -n_2\sin\omega + n_1n_3(1-\cos\omega) \\ -n_3\sin\omega + n_1n_2(1-\cos\omega) & \cos\omega + n_2^2(1-\cos\omega) & n_1\sin\omega + n_2n_3(1-\cos\omega) \\ n_2\sin\omega + n_1n_3(1-\cos\omega) & -n_1\sin\omega + n_2n_3(1-\cos\omega) & \cos\omega + n_3^2(1-\cos\omega) \end{pmatrix}$$

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{R(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}$$

$$\mathbf{u}(\omega,\hat{\mathbf{n}}) = \begin{pmatrix} \cos\frac{\omega}{2} + in_3\sin\frac{\omega}{2} & (in_1+n_2)\sin\frac{\omega}{2} \\ (in_1-n_2)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} - in_3\sin\frac{\omega}{2} \end{pmatrix}$$

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{\mathbf{u}(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} in_3 & in_1+n_2 \\ in_1-n_2 & -in_3 \end{pmatrix}$$

Irreducible representations of SO(3) and SU(2)
-- focusing on SU(2)

Consider the generator matrices

$$\mathbf{a} = \lim_{\omega\to 0}\left(\frac{\mathbf{u}(\omega,\hat{\mathbf{n}})-1}{\omega}\right) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} in_3 & in_1+n_2 \\ in_1-n_2 & -in_3 \end{pmatrix}$$

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$

In order to determine irreducible representations of SU(2), determine eigenstates associated with generators.

Eigenfunctions associated with generator functions of SU(2)

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$

It is convenient to map these generators into Hermitian matrices

$$\mathbf{A}_p \equiv -i\mathbf{a}_p = \tfrac{1}{2}\boldsymbol{\sigma}_p$$

$$\mathbf{A}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathbf{A}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Commutation relations: $[\mathbf{A}_1, \mathbf{A}_2] = i\mathbf{A}_3$

Define $\mathbf{A}^2 = \mathbf{A}_1^2 + \mathbf{A}_2^2 + \mathbf{A}_3^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{A}_- = \mathbf{A}_1 - i\mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_+ = \mathbf{A}_1 + i\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Eigenfunctions associated with generator functions of SU(2)

Relationships between operators:

$$[\mathbf{A}^2, \mathbf{A}_p] = 0$$

$$\mathbf{A}_-\mathbf{A}_+ = \mathbf{A}^2 - \mathbf{A}_3^2 - \mathbf{A}_3$$

$$\mathbf{A}_+\mathbf{A}_- = \mathbf{A}^2 - \mathbf{A}_3^2 + \mathbf{A}_3$$

Note that this "algebra" is identical to that of the total angular momentum operators

$$\mathbf{J}_x = \hbar\mathbf{A}_1 \quad \mathbf{J}_y = \hbar\mathbf{A}_2 \quad \mathbf{J}_z = \hbar\mathbf{A}_3$$

$\Rightarrow$ Leap to eigenstates of $\mathbf{J}^2$ and $\mathbf{J}_z$: $|jm\rangle$

$$\mathbf{A}^2 |jm\rangle = j(j+1)|jm\rangle$$

$$\mathbf{A}_3 |jm\rangle = m|jm\rangle$$

Eigenfunctions associated with generator functions of SU(2)

Additional relationships:

$$\mathbf{A}_- |jm\rangle = \{(j+m)(j-m+1)\}^{1/2} |j(m-1)\rangle$$

$$\mathbf{A}_+ |jm\rangle = \{(j+m)(j+m+1)\}^{1/2} |j(m+1)\rangle$$

Mapping the representations back to the original generators of SU(2): $\mathbf{a}_p = i\mathbf{A}_p$

$$D^j_{mm'}(\mathbf{a}_p) \equiv \langle jm|\mathbf{a}_p|jm'\rangle$$

$j = 0$ $\quad D^j_{mm'}(\mathbf{a}_p) = 0$

$j = \frac{1}{2}$ $\quad D^j_{mm'}(\mathbf{a}_1) = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad D^j_{mm'}(\mathbf{a}_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad D^j_{mm'}(\mathbf{a}_3) = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Mapping the representations back to the original generators of SU(2): $\mathbf{a}_p = i\mathbf{A}_p$

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$$D^j_{mm'}(\mathbf{a}_3)=\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

$$j=1 \qquad D^j_{mm'}(\mathbf{a}_1)=i\,\frac{\sqrt{2}}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad D^j_{mm'}(\mathbf{a}_2)=\frac{\sqrt{2}}{2}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}$$

$$D^j_{mm'}(\mathbf{a}_3)=i\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

Eigenfunctions associated with generator functions of SO(3)

$$R(\omega,\hat{\mathbf{n}})=\begin{pmatrix}\cos\omega+n_1^2(1-\cos\omega) & n_3\sin\omega+n_1n_2(1-\cos\omega) & -n_2\sin\omega+n_1n_3(1-\cos\omega)\\ -n_3\sin\omega+n_1n_2(1-\cos\omega) & \cos\omega+n_2^2(1-\cos\omega) & n_1\sin\omega+n_2n_3(1-\cos\omega)\\ n_2\sin\omega+n_1n_3(1-\cos\omega) & -n_1\sin\omega+n_2n_3(1-\cos\omega) & \cos\omega+n_3^2(1-\cos\omega)\end{pmatrix}$$

$$\mathbf{a}=\lim_{\omega\to 0}\left(\frac{R(\omega,\hat{\mathbf{n}})-1}{\omega}\right)=n_1\mathbf{a}_1+n_2\mathbf{a}_2+n_3\mathbf{a}_3=\begin{pmatrix}0&n_3&-n_2\\-n_3&0&n_1\\n_2&-n_1&0\end{pmatrix}$$

$$\mathbf{a}_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} \qquad \mathbf{a}_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix} \qquad \mathbf{a}_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

$$[\mathbf{a}_1,\mathbf{a}_2]=-\mathbf{a}_3$$

$$\mathbf{a}_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} \qquad \mathbf{a}_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix} \qquad \mathbf{a}_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

Commutation relations: $[\mathbf{a}_1,\mathbf{a}_2]=-\mathbf{a}_3$

It is convenient to map these generators into Hermitian matrices

$\mathbf{A}_p \equiv -i\mathbf{a}_p$

$$\mathbf{A}_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \qquad \mathbf{A}_2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix} \qquad \mathbf{A}_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

Commutation relations: $[\mathbf{A}_1,\mathbf{A}_2]=i\mathbf{A}_3$

Define $\mathbf{A}^2=\mathbf{A}_1^2+\mathbf{A}_2^2+\mathbf{A}_3^2=2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Consistent with representation derived from SU(2) analysis:

$$D^j_{mm'}(\mathbf{a}_p) \equiv \langle jm|\mathbf{a}_p|jm'\rangle$$

$$j=0 \qquad D^j_{mm'}(\mathbf{a}_p)=0$$

$$j=\tfrac{1}{2} \qquad D^j_{mm'}(\mathbf{a}_1)=\frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \qquad D^j_{mm'}(\mathbf{a}_2)=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$D^j_{mm'}(\mathbf{a}_3)=\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

$$j=1 \qquad D^j_{mm'}(\mathbf{a}_1)=i\,\frac{\sqrt{2}}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad D^j_{mm'}(\mathbf{a}_2)=\frac{\sqrt{2}}{2}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$

$$D^j_{mm'}(\mathbf{a}_3)=i\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Relationship to differential forms:

For each generator $\mathbf{a}_p$:

$$\mathbf{a}_p = \lim_{t\to 0}\left(\frac{e^{t\mathbf{a}_p t}-1}{t}\right)$$

Consider $$\lim_{t\to 0}\left(\frac{e^{t\mathbf{a}_p t}-1}{t}\right) f(\mathbf{r}) = \lim_{t\to 0}\left(\frac{f\left(\left(1-t\mathbf{a}_p\right)\mathbf{r}\right)-f(\mathbf{r})}{t}\right)$$

$$\approx -\mathbf{r}\mathbf{a}_p \nabla f(\mathbf{r})$$