PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 2:

Representation Theory

Reading: Chapter 2 in DDJ

1. **Review of group definitions**
2. **Theory of representations**

PHY 745 Group Theory

MWF 11-11:50 AM	OPL 102	http://www.wfu.edu/~natalie/s17phy745/

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Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/18/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		01/18/2017
	Mon: 01/16/2017		*MLK Holiday – no class*		
3	Wed: 01/18/2017				
4	Fri: 01/20/2017				
5	Mon: 01/23/2017				

Group theory
An abstract algebraic construction in mathematics

Definition of a group:

A group is a collection of "elements" – $A, B, C, \ldots$ and a "multiplication" process. The abstract multiplication $(\cdot)$ pairs two group elements, and associates the "result" with a third element. (For example $(A \cdot B = C)$.) The elements and the multiplication process must have the following properties.

1. The collection of elements is closed under multiplication. That is, if elements A and B are in the group and $A \cdot B = C$, element C must be in the group.
2. One of the members of the group is a "unit element" (E). That is, for any element A of the group, $A \cdot E = E \cdot A = A$.
3. For each element A of the group, there is another element A^{-1} which is its "inverse". That is $A \cdot A^{-1} = A^{-1} \cdot A = E$.
4. The multiplication process is "associative". That is for sequential mulplication of group elements A, B, and C, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Some definitions:

Order of the group → number of elements (members) in the group (positive integer for finite group, ∞ for infinite group)

Subgroup → collection of elements within a group which by themselves form a group

Coset → Given a subgroup $\mathbf{g}_\mathbf{i}$ of a group a right coset can be formed by multiply an element of $\mathbf{g}$ with each element of $\mathbf{g}_\mathbf{i}$

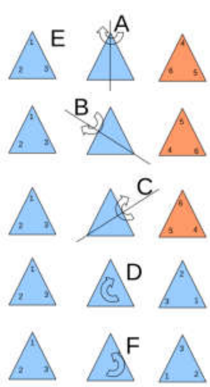
Class → members of a group generated by the conjugate construction $\mathcal{C} = X_i^{-1} Y X_i$ where Y is a fixed group element and X_i are all of the elements of the group.

Example of a 6-member group *E,A,B,C,D,F,G*

Group multiplication table

Group of order 6

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D



	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Subgroups:

E $\quad (E,A)$ $\quad (E,D,F)$

(E,B)

(E,C)

Classes:

$\mathcal{C}_1 = E$

$\mathcal{C}_2 = A, B, C$

$\mathcal{C}_3 = D, F$

Example of cyclic group of order 4:

	E	A	A^2	A^3
E	E	A	A^2	A^3
A	A	A^2	A^3	E
A^2	A^2	A^3	E	A
A^3	A^3	E	A	A^2

Subgroups:

$E \qquad (E, A^2)$

Classes:

$\mathcal{C}_1 = E$

$\mathcal{C}_2 = A$

$\mathcal{C}_3 = A^2$

$\mathcal{C}_4 = A^3$

Example of non-cyclic group of order 4

	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Subgroups:

$E \qquad (E, A)$

(E, B)

(E, C)

Classes:

$\mathcal{C}_1 = E$

$\mathcal{C}_2 = A$

$\mathcal{C}_3 = B$

$\mathcal{C}_4 = C$

Representations of a group

A representation of a group is a set of matrices (one for each group element) -- $\Gamma(A), \Gamma(B)$... that satisfies the multiplication table of the group. The dimension of the matrices is called the dimension of the representation.

Example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Note that the one-dimensional "identical representation"

$\Gamma^1(A) = \Gamma^1(B) = \Gamma^1(C) = \Gamma^1(D) = \Gamma^1(E) = \Gamma^1(F) = 1$ is always possible

Another one-dimensional representation is

$$\Gamma^2(A) = \Gamma^2(B) = \Gamma^2(C) = -1$$

$$\Gamma^2(E) = \Gamma^2(D) = \Gamma^2(F) = 1$$

Example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

A two-dimensional representation is

$$\Gamma^3(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma^3(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Gamma^3(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma^3(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

What about 3 or 4 dimensional representations for this group?

For example, the following 3 dimensional representation satisfies the multiplication table:

$$\Gamma(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \Gamma(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma(C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma(D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma(F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Reduced form of representation

$$\Gamma(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad \Gamma(A)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\quad \Gamma(B)=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$$

$$\Gamma(C)=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\quad \Gamma(D)=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}\quad \Gamma(F)=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$$

In this example, the "irreducible" representations for the group are 2 one-dimensional and 1 two-dimensional

Comment about representation matrices

A representation is not fundamentally altered by a similarity transformation

$\Gamma'(A)=S^{-1}\Gamma(A)S$

Check: Suppose $\Gamma(AB)=\Gamma(A)\Gamma(B)$

$$\Gamma'(AB)=S^{-1}\Gamma(AB)S=S^{-1}\Gamma(A)\Gamma(B)S$$
$$=S^{-1}\Gamma(A)SS^{-1}\Gamma(B)S$$
$$=\Gamma'(A)\Gamma'(B)$$

- Typically, unitary matrices are chosen for representations — unitary matrix: $UU^{\dagger}=U^{\dagger}U=1$
- Typically representations are reduced to block diagonal form and the irreducible blocks are considered in the representation theory

$$\Gamma(A)=\begin{pmatrix}x&y&\ldots&z\\y^*&w&\ldots&u\\\vdots&\vdots&\vdots&\vdots\\z^*&u^*&\ldots&w\end{pmatrix}\Rightarrow$$

similarity transformation

block diagonal form

$$\begin{pmatrix}\Gamma^1(A)&0&0&0\\0&\Gamma^2(A)&0&0\\0&0&\ddots&0\\0&0&0&\Gamma^n(A)\end{pmatrix}$$

distinct irreducible representation

The great orthogonality theorem

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\mu\nu\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

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Great orthogonality theorem continued

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Analysis shows that

$$\sum_i l_i^2 = h$$

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Simplified analysis in terms of the "characters" of the representations

$$\chi^j(R) \equiv \sum_{\mu=1}^{l_j} \Gamma^j(R)_{\mu\mu}$$

Character orthogonality theorem

$$\sum_R \left(\chi^i(R)\right)^* \chi^j(R) = h\delta_{ij}$$

Note that all members of a class have the same character for any given representation *i.*

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What the great ("wonderful") orthogonality theorem will do for us:

1. Show that there are a fixed number of distinct irreducible representations and help us find them
2. Show that the irreducible representations of a group have properties of orthogonal vector spaces.
3. Result in a simplified orthogonality theorem based on the "characters" of the group

Often the irreducible representations are related to physical quantities such as quantum mechanical wavefunctions or operators.

Example similarity transformation

$$\text{Let} \quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \quad S^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$S^{-1}MS = S^{-1}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\theta} & \frac{1}{\sqrt{2}}e^{i\theta} \\ \frac{i}{\sqrt{2}}e^{-i\theta} & \frac{-i}{\sqrt{2}}e^{i\theta} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Note: Two representations $\Gamma^i(R)$ and $\Gamma^i{}'(R)$ are equivalent if $\Gamma^i{}'(R) = S^{-1}\Gamma^i(R)S$